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Geometry of spectral curves and all order dispersive integrable system

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Abstract

We propose a definition for the tau function and a Baker-Akhiezer spinor kernel of an integrable system whose times parametrize slow deformations (at a speed $1/N$) of a given algebraic plane curve. This definition is a full asymptotic series in $1/N$ involving theta functions. The large N limit of this construction reproduces the algebro-geometric solutions of the multi–KP equation. We check that our tau function satisfies Hirota equations to the first two orders, and we conjecture that they hold to all orders. The Hirota equations are equivalent to a self-replication property for the Baker-Akhiezer spinor kernel. We analyse its consequences, namely the reconstruction of a isomonodromic problem given by a Lax pair, and the relation between "correlators", the tau function and the Baker-Akhiezer spinor kernel. This construction highlights the bridges between symplectic invariants, integrable hierarchies and enumerative geometry.

1 Introduction to integrability

Integrable systems play a particularly important role in physics, because they are "exactly solvable", at least in principle. Under a suitable change of variables, they can be brought into the form of a linear constant motion with constant velocity in a multi-dimensional torus. However, there is a difference between exactly solvable and exactly solved, and the study of integrable systems is still a very active branch of theoretical physics and mathematical physics. See the books [DJM00, BBT02] for an introduction.

There exists strong links between geometry and integrability [Kri77, SS83, SW85, Hit87, Man88, Dub92, Kri94, IKF95]. Besides, the interplay with algebraic geometry

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is an important issue. Indeed, generating functions which encode the enumerative geometry in various kinds of complex manifolds have been shown to satisfy integrable equations. It happens for instance for generating functions for intersection numbers [Wit91, Kon92, Giv03], Hurwitz numbers and Hodge integrals [Oko00, Kaz09], or Gromov-Witten invariants in toric Calabi-Yau threefolds [ADK+06, Bri11].

There are several definition of what is an integrable system. In this article, we speak of an integrable system whenever some Hirota equations [Hir71] are present. An important class of integrable systems arise from Lax systems. The locus of eigenvalues of the Lax operator is an algebraic curve called the spectral curve. Here, conversely, we shall deal with the ways to build an integrable system (and in fact a Lax pair) starting from an arbitrary algebraic spectral curve, the so-called reconstruction problem. This has been addressed, in a dispersionless limit, in several earlier works (especially [Kri77] and [BG07]), on which we will comment along this article. Here, we propose a way to add the dispersionful part, using the algebraic invariants introduced in [EO07a].

Let us recall first generalities about Lax pairs and integrable systems.

1.1 Hamiltonian system and Liouville integrability

Consider a classical Hamiltonian system, i.e. a set of coordinates $q_i(t)$ and their conjugated momenta $p_i(t)$, obeying the Hamilton equations of motions:

$$\frac{d}{dt}\,p_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{d}{dt}\,q_i = \frac{\partial H}{\partial p_i} \tag{1.1}$$

where H is the Hamiltonian. This can be written in vector notation:

$$\vec{Q} = (q_1,\dots,q_n,p_1\dots,p_n), \qquad \frac{d\vec{Q}}{dt} = \{H,\vec{Q}\} \tag{1.2}$$

where $\{,\}$ is the Poisson bracket such that $\{p_i,q_j\} = \delta_{i,j}$.

The system is "Liouville integrable" if there exist n independent conserved quantities $H_1,H_2,\dots,H_n$ (by convention $H_1 = H$):

$$\{H_i,H_j\} = 0 \tag{1.3}$$

In this case, it is possible to find a symplectic change of coordinates in the phase space, called "action-angle" variables, in which the equations of motions reduce to linear constant motions in a torus (the Jacobian, i.e. u_i is a periodic variable):

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = v_i, \qquad \frac{\mathrm{d}v_i}{\mathrm{d}t} = 0 \tag{1.4}$$

The main problem is to find explicitly the conserved quantities H_i, the action variables v_i, and the angle variables u_i as functions of the initial system of coordinates q_i,p_i.

A way to achieve this, was found in terms of a Lax pair, which we review below, and all the quantities above can be expressed in terms of algebraic geometry.

1.2 Lax matrix formulation

Imagine that one can group the coordinates q_i, p_j together, using d^2 formal series $L_{i,j}(x,\vec{Q})$, $i,j=1,\dots,d$ of a formal variable x, which form a $d\times d$ matrix $L(x,\vec{Q})$. This means that the coefficients of the expansion of $L_{i,j}(x,\vec{Q})$ in powers of x, are functions of the coordinates q,p. Imagine that there exists a matrix $M(x,\vec{Q})$, such that the equations of motion Eqn. 1.2 can be rewritten as a matrix equation:

$$\frac{d}{dt}L(x,\vec{Q}(t)) = [L(x,\vec{Q}(t)), M(x,\vec{Q}(t))] \tag{1.5}$$

This equation is called a Lax equation, L is called the Lax matrix, x is called the spectral parameter, and (L,M) is called a Lax pair. It is easy to see that the Lax equation implies that for any x, the eigenvalues of $L(x,\vec{Q}(t))$ are conserved quantities, and by expansion into powers of x, we obtain all conserved quantities.

It may not seem obvious that the Hamilton equations of motion Eqn. 1.2 of a Liouville integrable system can always be written in this way, but indeed a Lax pair[3] has been found for all classical integrable systems (see for example [BBT02] and references thereof).

1.3 The spectral curve

The Lax equation (Eqn. 1.5) implies that the eigenvalues of $L(x,\vec{Q}(t))$ do not depend on the time t, they are conserved. In particular, this means that the characteristic polynomial $\mathcal{E}(x,y)$ is independent of time:

$$\mathcal{E}(x,y) = \det(y\ \mathrm{id} - L(x,\vec{Q}(t))) \tag{1.6}$$

The locus of zeroes of $\mathcal{E}(x,y)$, i.e. the eigenvalues of L, is called the "spectral curve" $\mathcal{S}$:

$$\mathcal{S} = \{(x,y)\,|\,\mathcal{E}(x,y)=0\} \tag{1.7}$$

By expanding $\mathcal{E}(x,y)$ into powers of x and y, one can find all the conserved quantities, and thus, knowing the spectral curve, one knows the conserved quantities. The non-conserved quantities, i.e. the angle variables, are contained in the eigenvectors of L. It was shown [Kri77, BBT02], that the eigenvectors of L, must be Baker-Akhiezer functions on the spectral curve. Before explaining what it means, we need to describe more geometry of the spectral curve, i.e. more algebraic geometry.

[3] Given a Hamiltonian system, there is neither existence, nor unicity, of an equivalent Lax system.

1.4 Outline of the article

- In Section 2, we present the algebro-geometric background needed for our purposes.
- We first review in Sections 3-4 the construction of an integrable system associated to a given compact Riemann surface. Its times are the moduli of a meromorphic 1-form defined on the curve with fixed cycle integrals. The conserved quantities are the moduli of the curve. This construction dates back to Krichever [Kri77]. It encompasses the finite gap approach when the spectral curve is hyperelliptic, and produces in general the so-called algebro-geometric solutions of KP. We slightly reformulate this construction by defining a "spinor kernel" $\psi_{\mathrm{cl}}(z_1, z_2)$, globally defined on the spectral curve (Section 3.1). Baker-Akhiezer functions and the integrable kernel can be recovered from ψ_{cl}. We recall the construction of the Tau function $\mathcal{T}_{\mathrm{cl}}$ and its properties. This integrable system is called "classical", or "dispersionless".
- The goal of this article is to propose explicit formulas for another type of integrable system, called "dispersive" [Dub96]. The times are now (not necessarily independent) moduli of a spectral curve, which evolve at speed of order $1/N$ around a given algebraic spectral curve $\mathcal{S}$. We define a function $\mathcal{T}$ which is an ad hoc solution to "loop equations" (Section 5), a spinor kernel $\psi(z_1, z_2)$ via a Sato-like formula (Section 5.3), and correlators $W_n(z_1, \ldots, z_n)$ encoding variations of $\mathcal{T}$ with respect to all possible moduli (Section 6). We present the conjecture that ψ is self-replicating, and see its consequences: Hirota equations for $\mathcal{T}$ (Section 7), determinantal formulas for the correlators, Christoffel-Darboux formula for the integrable kernel, and a Lax pair formulation (Section 8).
- Eventually, in Section 9, we recall how the results known about asymptotics of matrix integrals are compatible with special cases of our construction.

2 Geometry of the spectral curve

See [FK07, Fay70] for an introduction to algebraic geometry.

2.1 Definition

For us, a spectral curve $\mathcal{S}$ is the data of a complex curve $\mathcal{C}$ and two meromorphic functions X, Y on $\mathcal{C}$:

$$\mathcal{S} = (\mathcal{C}, X, Y) \tag{2.1}$$

An essential example is given by the zero locus of a polynomial:

$$\mathcal{E}(x,y) = 0 \tag{2.2}$$

If $\deg_y \mathcal{E} = d$, then for every value x, there are d values (with multiplicities) $y = y_1(x), \ldots, y_d(x)$ solutions of that equation. Those d functions can also be viewed together, as d branches of one multivalued[4] function $x \mapsto y(x)$. The equation $\mathcal{E}(x,y) = 0$ can also be seen as the embedding of a Riemann surface $\mathcal{C}$ into $\mathbb{C} \times \mathbb{C} \cup \{\infty\}$. In other words, we prefer to represent the spectral curve in a parametric form:

$$\mathcal{S} = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid \mathcal{E}(x,y) = 0\} = \{(X(z), Y(z)) \ , \ z \in \mathcal{C}\} \tag{2.3}$$

This means that there are two analytical functions (meromorphic) X and Y on $\mathcal{C}$, which generate $\mathcal{S}$.

In this article, we only work with spectral curves for which $\mathcal{C}$ is compact. Then, there exists a polynomial $\mathcal{E}$ such that $\mathcal{E}(X,Y) = 0$. We also assume $\mathcal{C}$ connected without losing generality.

2.2 Some notations and properties

2.2.1 Topology and holomorphic 1-forms

The curve $\mathcal{C}$ is either simply connected, and then this is the Riemann sphere $\mathcal{C} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, or it has genus $\mathfrak{g} > 0$. Then, any maximal open contractible subset of $\mathcal{C}$ is called a fundamental domain. If it is of genus $\mathfrak{g} > 0$, there exist $2\mathfrak{g}$ independent non-contractible cycles (see Fig. 1), and we can choose them in such a way (but not unique) that:

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \qquad \mathcal{A}_i \cap \mathcal{A}_j = 0, \qquad \mathcal{B}_i \cap \mathcal{B}_j = 0 \tag{2.4}$$

A basis satisfying these intersection relations is called "symplectic".

From the topological point of view, a genus $\mathfrak{g} > 0$ compact Riemann surface with a basis $(\mathcal{A}_i, \mathcal{B}_i)_{1 \leq i \leq \mathfrak{g}}$ is a $4\mathfrak{g}$ closed polygon Γ, with edges

$$[\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_1^{-1}, \mathcal{B}_1^{-1}, \ldots, \mathcal{A}_\mathfrak{g}, \mathcal{B}_\mathfrak{g}, \mathcal{A}_\mathfrak{g}^{-1}, \mathcal{B}_\mathfrak{g}^{-1}] \tag{2.5}$$

glued by pairs. $\mathring{\Gamma}$ is a fundamental domain of $\mathcal{C}$. It is a classical result, that on a curve of genus $\mathfrak{g}$, there exists $\mathfrak{g}$ independent holomorphic 1-forms $\mathrm{d}u_i$ (holomorphic means in particular having no poles), and they can be normalized on the $\mathcal{A}$-cycles:

$$\oint_{\mathcal{A}_i} \mathrm{d}u_j = \delta_{i,j} \tag{2.6}$$

[4]We could also consider the multivalued function $y \mapsto x(y)$, and indeed, symplectic invariance guarantees that this would give the same Tau function.

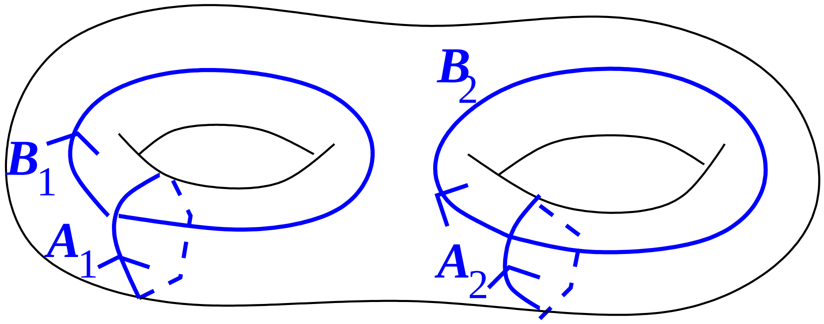

Figure 1: A symplectic basis of $2\mathfrak{g}$ non-contractible cycles on a Riemann surface of genus $\mathfrak{g}$.

Then, the $\mathfrak{g} \times \mathfrak{g}$ matrix $\tau_{i,j}$:

$$\tau_{i,j} = \oint_{\mathcal{B}_i} \mathrm{d}u_j \tag{2.7}$$

is known to be symmetric $\tau_{i,j} = \tau_{j,i}$ and its imaginary part is definite positive:

$$\tau^t = \tau, \qquad \mathrm{Im}\,\tau > 0 \tag{2.8}$$

τ is called the Riemann matrix of periods of $\mathcal{C}$.

2.2.2 Riemann bilinear identity

It is well-known that, for any meromorphic 1-form ω defined on $\mathcal{C}$:

$$\sum_{p\,\text{pole of}\,\omega} \operatorname*{Res}_p \ \omega = 0. \tag{2.9}$$

This can be generalized to some forms which are not globally defined on $\mathcal{C}$. Let ω_1, ω_2 be two meromorphic 1-form defined on $\mathcal{C}$, such that $\mathrm{Res}\,\omega_2 = 0$ for every pole of ω_2, and let ϕ_2 a primitive of ω_2 defined on a fundamental domain of $\mathcal{C}$, i.e. $\mathrm{d}\phi_2 = \omega_2$. For any symplectic basis of homology $(\mathcal{A}, \mathcal{B})$, we have the Riemann bilinear identity:

$$\sum_{p\,\text{pole of}\,(\phi_2\omega_1)} \operatorname*{Res}_p \ \phi_2\omega_1 = \frac{1}{2i\pi} \sum_{i=1}^{\mathfrak{g}} \left(\oint_{\mathcal{A}_i} \omega_1 \cdot \oint_{\mathcal{B}_i} \omega_2 - \oint_{\mathcal{B}_i} \omega_1 \cdot \oint_{\mathcal{A}_i} \omega_2 \right). \tag{2.10}$$

2.2.3 Theta functions

Given any symmetric matrix τ such that $\mathrm{Im}\,\tau > 0$, one can define the Riemann Theta function:

$$\theta(\mathbf{u}|\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^{\mathfrak{g}}} \mathrm{e}^{2i\pi \mathbf{n}\cdot\mathbf{u}}\, \mathrm{e}^{i\pi\, \mathbf{n}^t\cdot\tau\cdot\mathbf{n}}. \tag{2.11}$$

Since $\mathrm{Im}\,\tau > 0$, it is a well-defined convergent series for all $\mathbf{u}$ in $\mathbb{C}^{\mathfrak{g}}$. Most often we will not write the τ dependence of θ:

$$\theta(\mathbf{u}|\tau) \equiv \theta(\mathbf{u}) \tag{2.12}$$

The Theta function has nice quasi-periodicity properties. If $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^{\mathfrak{g}}$ we have:

$$\theta(\mathbf{u}+\mathbf{n}+\tau\mathbf{m}) = \mathrm{e}^{-i\pi\,(2\,\mathbf{m}^t\cdot\mathbf{u}+\mathbf{m}^t\cdot\tau\cdot\mathbf{m})}\;\theta(\mathbf{u}) \tag{2.13}$$

It also satisfies the heat equation:

$$\partial_{\tau_{i,j}}\,\theta = \frac{1}{4i\pi}\,\partial_{u_i}\partial_{u_j}\,\theta \tag{2.14}$$

In this equation, $\tau_{i,j}$ and $\tau_{j,i}$ are considered independent.

2.2.4 Jacobian and Abel map

Let us choose a generic basepoint $o \in \mathcal{C}$, which will in fact play no role. For any point $z \in \mathcal{C}$, we define:

$$\forall\, i \in \{1,\ldots,\mathfrak{g}\}\,, \qquad u_i(z) = \int_o^z \mathrm{d}u_i \tag{2.15}$$

where the integration path is chosen such that it does not intersect any $\mathcal{A}$-cycle or $\mathcal{B}$-cycle. Then we define the vector:

$$\mathbf{u}(z) = [u_i(z)]_{1\leq i\leq \mathfrak{g}}\ \in \mathbb{C}^{\mathfrak{g}} \tag{2.16}$$

The application $z \mapsto \mathbf{u}(z) \bmod (\mathbb{Z}^{\mathfrak{g}} + \tau\mathbb{Z}^{\mathfrak{g}})$ is well-defined and analytical, it maps the spectral curve into the Jacobian $\mathbb{J} = \mathbb{C}^{\mathfrak{g}}/(\mathbb{Z}^{\mathfrak{g}} + \tau\mathbb{Z}^{\mathfrak{g}})$. This defines the Abel map:

$$\begin{aligned} &\mathcal{C} \to \mathbb{J} \\ &z \mapsto \mathbf{u}(z) \bmod (\mathbb{Z}^{\mathfrak{g}} \oplus \tau\mathbb{Z}^{\mathfrak{g}}) \end{aligned} \tag{2.17}$$

The Jacobi inversion theorem states that every $\mathbf{w} \in \mathbb{J}$ can be represented as $\mathbf{w} = \sum_{j=1}^{\mathfrak{g}} \mathbf{u}(p_j)$ for some points $p_1,\ldots,p_{\mathfrak{g}} \in \mathcal{C}$.

The Theta function can be used with τ the Riemann matrix of periods of a Riemann surface $\mathcal{C}$, and $\mathbf{u}$ the Abel map of a point on $\mathcal{C}$. In this case, it enjoys other important properties. Its zero locus has the following description: there exists $\mathbf{k} \in \mathbb{C}^{\mathfrak{g}}$, so that $\theta(\mathbf{w}|\tau) = 0$ iff there exists $\mathfrak{g}-1$ points $z_1,\ldots,z_{\mathfrak{g}-1} \in \mathcal{C}$ satisfying $\mathbf{w} = \sum_{j=1}^{\mathfrak{g}-1}\mathbf{u}(z_j) + \mathbf{k}$. $\mathbf{k}$ is called a "Riemann vector of constants", and it depends on the basepoint o used to define the Abel map $\mathbf{u}$.

2.2.5 Prime form

An odd characteristics $\mathbf{c}$ is a vector of the form:

$$\mathbf{c} = \frac{\mathbf{n}+\tau\mathbf{m}}{2}, \qquad \mathbf{n},\mathbf{m}\in\mathbb{Z}^{\mathfrak{g}}, \qquad \mathbf{n}^t\cdot\mathbf{m}\in 2\mathbb{Z}+1 \tag{2.18}$$

The Theta function vanishes at odd characteristics: $\theta(\mathbf{c}) = 0$, and the following holomorphic form:

$$\mathrm{d}h_{\mathbf{c}}(z) = \sum_{i=1}^{\mathfrak{g}} \mathrm{d}u_i(z)\,\partial_{u_i}\theta(\mathbf{c}) \tag{2.19}$$

has only double zeroes on $\mathcal{C}$, so that we can define its squareroot, and thus one can define the prime form:

$$E(z_1,z_2) = \frac{\theta(\mathbf{u}(z_1)-\mathbf{u}(z_2)+\mathbf{c})}{\sqrt{\mathrm{d}h_{\mathbf{c}}(z_1)\;\mathrm{d}h_{\mathbf{c}}(z_2)}} \tag{2.20}$$

There exists choices of $\mathbf{c}$ such that E is not identically 0 (we say $\mathbf{c}$ is "non singular"), and E is in fact independent of such $\mathbf{c}$. It is a $(-1/2,-1/2)$-form on $\mathcal{C}$, and it vanishes only at $z_1 = z_2$. In any local coordinate $\xi(z)$ we have:

$$E(z_1,z_2) \underset{z_1\to z_2}{=} \frac{\xi(z_1)-\xi(z_2)}{\sqrt{\mathrm{d}\xi(z_1)\,\mathrm{d}\xi(z_2)}} + O((\xi(z_1)-\xi(z_2))^3) \tag{2.21}$$

Because of the Theta function, $E(z_1,z_2)$ is not a global $(-1/2,-1/2)$ form on $\mathcal{C}$. It transforms according to Eqn. 2.13.

The Theta function associated to a Riemann surface satisfies a non-linear relation called Fay identity [Fay70]: for any $z_1,z_2,z_3,z_4\in\mathcal{C}$, any $\mathbf{w}\in\mathbb{C}^{\mathfrak{g}}$,

$$\begin{aligned} &\theta(\mathbf{w}+\mathbf{c})\theta(\mathbf{u}_{12}+\mathbf{u}_{34}+\mathbf{w}+\mathbf{c})\,\frac{E(z_1,z_3)E(z_2,z_4)}{E(z_1,z_4)E(z_2,z_3)}\,\frac{1}{E(z_1,z_2)E(z_3,z_4)} \\ =\ & \frac{\theta(\mathbf{w}+\mathbf{u}_{12}+\mathbf{c})}{E(z_1,z_2)}\frac{\theta(\mathbf{w}+\mathbf{u}_{34}+\mathbf{c})}{E(z_3,z_4)} - \frac{\theta(\mathbf{w}+\mathbf{u}_{14}+\mathbf{c})}{E(z_1,z_4)}\frac{\theta(\mathbf{w}+\mathbf{u}_{32}+\mathbf{c})}{E(z_3,z_2)} \end{aligned} \tag{2.22}$$

where $\mathbf{u}_{jl} = \mathbf{u}(z_j)-\mathbf{u}(z_l)$.

2.2.6 Bergman kernel

The Bergman kernel [BS53] is the "fundamental (1,1)-form of the second kind" [Fay70], defined as:

$$B(z_1,z_2) = \mathrm{d}_{z_1}\mathrm{d}_{z_2}\ln\left(\theta(\mathbf{u}(z_1)-\mathbf{u}(z_2)+\mathbf{c})\right) \tag{2.23}$$

It is independent of the choice of a non-singular, odd characteristics $\mathbf{c}$. It is a globally defined, symmetric (1,1)-form, having a double pole at $z_1 = z_2$ with no residue, and no other pole. It is normalized so that:

$$\oint_{\mathcal{A}_i} B(\cdot,z) = 0, \qquad \oint_{\mathcal{B}_i} B(\cdot,z) = 2i\pi\,\mathrm{d}u_i(z) \tag{2.24}$$

Near $z_1 = z_2$, it behaves, in any local coordinate $\xi(z)$, like:

$$B(z_1, z_2) \underset{z_1 \to z_2}{=} \frac{\mathrm{d}\xi(z_1)\, \mathrm{d}\xi(z_2)}{(\xi(z_1) - \xi(z_2))^2} + O(1) \tag{2.25}$$

We also define the fundamental 1-form of the third kind:

$$\mathrm{d}S_{z_1,z_2}(z) = \int_{z_2}^{z_1} B(\cdot, z) \tag{2.26}$$

where the integration contour is chosen so that it does not intersect any $\mathcal{A}$-cycle or $\mathcal{B}$-cycle. It is a 1-form in the variable z, and a function of the variable z_1, z_2, and it satisfies:

$$\oint_{\mathcal{A}_j} \mathrm{d}S_{z_1,z_2} = 0 \qquad , \qquad \oint_{\mathcal{B}_j} \mathrm{d}S_{z_1,z_2} = 2i\pi(u_j(z_1) - u_j(z_2)) \tag{2.27}$$

It has a simple pole at $z = z_1$ with residue $+1$, a simple pole at $z = z_2$ with residue -1, and no other pole. In other words, in any local coordinate $\xi(z)$:

$$\mathrm{d}S_{z_1,z_2}(z) \underset{z \to z_1}{\sim} \frac{\mathrm{d}\xi(z)}{\xi(z) - \xi(z_1)} \tag{2.28}$$

$$\mathrm{d}S_{z_1,z_2}(z) \underset{z \to z_2}{\sim} \frac{-\,\mathrm{d}\xi(z)}{\xi(z) - \xi(z_2)} \tag{2.29}$$

Notice that in the variable z it is globally defined for $z \in \mathcal{C}$ (it has no monodromy if z goes around a non-contractible cycle), whereas in the variable z_1 (resp. z_2) it is defined only on the fundamental domain, it has monodromies when z_1 (resp. z_2) goes around a non-contractible cycle $\mathcal{B}_j$:

$$\mathrm{d}S_{z_1+\mathcal{B}_j,z_2}(z) = \mathrm{d}S_{z_1,z_2}(z) + 2i\pi\, du_j(z), \tag{2.30}$$

resp.

$$\mathrm{d}S_{z_1,z_2+\mathcal{B}_j}(z) = \mathrm{d}S_{z_1,z_2}(z) - 2i\pi\, du_j(z). \tag{2.31}$$

2.3 Parametrization of meromorphic forms

2.3.1 Sheets, ramification and branchpoints, local coordinate patches

If $\deg_y \mathcal{E} = d$ then $\mathcal{E}(x, y) = 0$ has d solutions, i.e., for every value x, there are d points $z^1(x), \ldots, z^d(x)$ on the curve $\mathcal{C}$ such that:

$$X(z^i(x)) = x \tag{2.32}$$

X is then a meromorphic function on $\mathcal{C}$ of degree d. $z^i(x)$ is sometimes called the preimage of x in the i-th sheet.

Definition 2.1 *We call "ramification points of order k", the zeroes of order $k \geq 1$ of the meromorphic* 1*-form* $\mathrm{d}X$*. If $a \in \mathcal{C}$ is a ramification point, the corresponding value $X(a)$ is called a branchpoint. All the other points $z \in \mathcal{C}$ at which $X(z)$ is analytical, are called "regular points".*

Definition 2.2 *We say that a branchpoint x_a is simple if $X^{-1}(\{x_a\})$ consists in $d-1$ points, one of them being a ramification point of order* 1*, and all the remaining ones being regular points.*

2.3.2 Definition of local ccordinates

Near a ramification point a of order k, $\xi_a(z) = (X - X(a))^{1/(k+1)}$ defines a local coordinate on the curve. Simple branchpoints play a special role in Sections 3.7, 5.1 and 6.1. For a simple branchpoint we have

$$\xi_a(z) = \sqrt{X(z) - X(a)}. \tag{2.33}$$

Since X is a meromorphic function of degree d, it has d poles with multiplicities, i.e.:

$$\infty_1^{d_{\infty_1}}, \ldots, \infty_s^{d_{\infty_s}}, \qquad \sum_i d_{\infty_i} = d \tag{2.34}$$

Near ∞_i, a good local variable is:

$$\xi_{\infty_i}(z) = X(z)^{-1/d_{\infty_i}} \tag{2.35}$$

Besides, we will need to consider also poles of a meromorphic form ω. If p is a pole of ω, but not a pole of X, neither a zero of $\mathrm{d}X$, a good local variable is:

$$\xi_p(z) = X(z) - X(p) \tag{2.36}$$

In this case, the multiplicity of p is $d_p = -1$. We shall now always use the local coordinates $\xi(z)$ defined above. Notice that they depend only on the function $X(z)$ and not on $Y(z)$.

Definition 2.3 *Given a meromorphic 1-form $\omega(z)$ which has no pole at ramification points, let us call:*

$$\begin{aligned} \mathcal{P} &= \{\text{poles of } \omega\} \\ \mathcal{P}_\infty &= \{\text{poles of } X\} \\ \overline{\mathcal{P}} &= \mathcal{P} \cup \mathcal{P}_\infty \end{aligned}$$

To any $p \in \overline{\mathcal{P}}$, we have associated a coordinate patch ξ_p on $\mathcal{C}$ centered on p.

2.3.3 Poles and times, filling fractions

Definition 2.4 *For any $p \in \overline{\mathcal{P}}$, we define the "times" near p as the coefficient of the negative part of the Laurent series expansion of ω near p:*

$$\omega(z) \underset{z\to p}{=} \sum_{j\geq 0} t_{p,j}\,(\xi_p(z))^{-(j+1)}\,\mathrm{d}\xi_p(z) + O(1)$$

i.e.

$$t_{p,j} = \operatorname*{Res}_{z\to p}\ \omega(z)\ \xi_p(z)^j$$

We also write collectively:

$$\vec{t}_p = [t_{p,j}]_{j\in\mathbb{N}}, \qquad \vec{t} = (\vec{t}_p)_{p\in\overline{\mathcal{P}}}$$

Notice that the times $t_{p,0} = \operatorname{Res}_p \omega$ are not independent, because the sum of residues of ω must vanish:

$$\sum_{p\in\overline{\mathcal{P}}} t_{p,0} = 0 \tag{2.37}$$

Definition 2.5 *We define the "filling fractions" (also called "conserved quantities") by:*

$$\epsilon_i = \frac{1}{2i\pi}\oint_{\mathcal{A}_i}\omega$$

2.3.4 Form-cycle duality

To each time t_k, we associate a differential meromorphic form $\omega_k(z)$, as well as a dual cycle ω_k^*, and a dual orthogonal cycle $\omega_k^{*\perp}$:

$$t_k \quad\leftrightarrow\quad \omega_k(z) \quad\leftrightarrow\quad \omega_k^* \quad\leftrightarrow\quad \omega_k^{*\perp} \tag{2.38}$$

In such a way that:

$$\left.\frac{\partial\,\omega(z)}{\partial t_k}\right|_{X(z)} = \omega_k(z), \qquad \omega(z) = \sum_k t_k\,\omega_k(z) \tag{2.39}$$

$$\omega_k(z) = \int_{\omega_k^*} B(\cdot,z), \qquad t_k = \int_{\omega_k^{*\perp}}\omega, \qquad \omega_i^*\cap\omega_j^{*\perp} = \delta_{i,j} \tag{2.40}$$

The symbol $|_{X(z)}$ means that we differentiate keeping the local coordinates ξ fixed (their definition depend only on the function X). More explicitly we have:

- Filling fractions $\epsilon_i \longrightarrow \omega_j =$ 1st kind differential $= 2i\pi du_j$:

  $\omega_j(z) = 2i\pi\,\mathrm{d}u_j(z) = \oint_{\mathcal{B}_j} B(z,\cdot)$,

  $\omega_j^* = \mathcal{B}_j$,

  $\omega_j^{*\perp} = \frac{1}{2i\pi}\mathcal{A}_j$.

- Residues $t_{p,0} \longrightarrow \omega_{p,0}$ = third kind differential:

  $\omega_{p,0}(z) = \mathrm{d}S_{p,o}(z) = \int_o^p B(z,\cdot)$,

  $\omega^*_{p,0} = [o,p]$,

  $\omega^{*\perp}_{p,0} = \frac{1}{2i\pi} C_p, \qquad t_{p,0} = \operatorname{Res}_p (Y \mathrm{d}X)$,

  where o is an arbitrary basepoint on $\mathcal{C}$, and C_p is a small circle surrounding p with index 1. As we mentioned, the $t_{p,0}$ are not independent variables, and only $(t_{p,0} - t_{p_0,0})_{p \neq p_0}$ for a fixed p_0 are independent. As a consequence, we see that only differences $\omega_{p,0} - \omega_{p',0}$ and $\omega^*_{p,0} - \omega^*_{p',0}$ are independent of a choice of basepoint o.

- Higher times $t_{p,j}$ with $j \geq 1 \longrightarrow \omega_{p,j}$ = second kind differential:

  $\omega_{p,j}(z) = B_{p,j}(z) = \operatorname{Res}_{z' \to p} \xi_p(z')^{-j} B(z',z)$,

  $\omega^*_{p,j} = \frac{1}{2i\pi} \xi_p^{-j} C_p$,

  $\omega^{*\perp}_{p,j} = \frac{1}{2i\pi} \xi_p^{j+1} C_p$.

The form ω is a linear combination of those meromorphic forms, and almost by definition we have:

Theorem 2.1

$$\omega(z) = \sum_k t_k \omega_k(z) = \sum_{i=1}^{\mathfrak{g}} 2i\pi\, \epsilon_i\, \mathrm{d}u_i(z) + \sum_{p \in \overline{\mathcal{P}}} t_{p,0}\, \mathrm{d}S_{p,o}(z) + \sum_{p \in \overline{\mathcal{P}},\, j \geq 1} t_{p,j}\, B_{p,j}(z)$$

We also define its dual cycle

$$\omega^* = \sum_k t_k \omega^*_k = \sum_{i=1}^{\mathfrak{g}} \epsilon_i\, \mathcal{B}_i + \sum_{p \in \overline{\mathcal{P}}} t_{p,0}\, [o,p] + \sum_{p \in \overline{\mathcal{P}},\, j \geq 1} t_{p,j}\, \omega^*_{p,j}.$$

2.4 The prepotential

The fact that $\int_{\omega^*_i} \int_{\omega^*_j} B(z,z')$ is symmetric, implies that there exists a function $F_0(\vec{t})$ called the "prepotential" such that:

$$\frac{\partial F_0}{\partial t_i} \text{ " = " } \int_{\omega^*_i} \omega, \qquad \frac{\partial^2 F_0}{\partial t_i \partial t_j} \text{ " = " } \int_{\omega^*_i} \int_{\omega^*_j} B \tag{2.41}$$

The problem (this is why we write quotation marks) is that those integrals are not well-defined for times associated to third kind differentials. Such a statement is correct after an appropriate regularization.

When z is in the vicinity of a pole p, we define:

$$V_p(z) = -\sum_{j \geq 1} \frac{t_{p,j}}{j}\, \xi_p(z)^{-j}, \qquad \mathrm{d}V_p(z) = \sum_{j \geq 1} t_{p,j}\, \frac{\mathrm{d}\xi_p(z)}{\xi_p(z)^{j+1}} \tag{2.42}$$

It is such that $\omega - \mathrm{d}V_p$ has at most a simple pole at p. Given an arbitrary base point $o \in \mathcal{C}$, the following integral is well-defined:

$$\mu_p = \int_o^{\infty_p} \left(\omega(z) - \mathrm{d}V_p(z) - t_{p,0}\,\frac{\mathrm{d}\xi_p(z)}{\xi_p(z)}\right) - V_p(o) - t_{p,0}\ln\xi_p(o) \tag{2.43}$$

μ_p depends on the base point o, but only by an additive constant independent of p. Since $\sum_p t_{p,0} = 0$, the sum $\sum_p t_{p,0}\,\mu_p$ is thus independent of o. In some sense, μ_p is a regularized version of $\int_{\omega^*_{p,0}} \omega$ (which does not exists). Since for all the other cycles, $\int_{\omega^*_k} \omega$ is well-defined, we can now define the prepotential.

Definition 2.6 *The prepotential is:*

$$F_0 = \frac{1}{2}\left[\sum_{p\in\overline{\mathcal{P}}} \operatorname*{Res}_p\, V_p\,\omega + \sum_{p\in\overline{\mathcal{P}}} t_{p,0}\,\mu_p + \sum_{i=1}^{\mathfrak{g}} \epsilon_i \oint_{\mathcal{B}_i} \omega\right]$$

Theorem 2.2 *The first derivatives of F_0 are given by, for $j \geq 1$:*

$$\frac{\partial F_0}{\partial t_{p,j}} = \oint_{\omega^*_{p,j}} \omega = \operatorname*{Res}_p\, \xi_p^{-j}\,\omega, \qquad \frac{\partial F_0}{\partial t_{p,0}} - \frac{\partial F_0}{\partial t_{p',0}} = \mu_p - \mu_{p'}$$

and:

$$\frac{\partial F_0}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} \omega$$

The proof of this theorem has appeared in many works, in particular [Dub92, Ber03, ?, Ber06, EO07a]. The definition of F_0 is thus equivalent to write

$$F_0 = \frac{1}{2}\sum_k t_k \frac{\partial F_0}{\partial t_k} \tag{2.44}$$

which means that F_0 is homogeneous of degree 2. Another classical result is:

Theorem 2.3 *The second derivatives of F_0 are given by:*

$$\frac{\partial^2 F_0}{\partial t_k \partial t_l} = \int_{\omega^*_k}\int_{\omega^*_l} B$$

except for the following cases:

$$\begin{aligned}
\frac{\partial}{\partial t_k}\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right) F_0 &= \int_{\omega^*_k}\int_{\omega^*_{p,0}} B - \int_{\omega^*_k}\int_{\omega^*_{p',0}} B \\
\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)^2 F_0 &= -\ln\left(E(p,p')^2\,\mathrm{d}\xi_p(p)\,\mathrm{d}\xi_{p'}(p')\right) \\
\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p'',0}}\right) F_0 &= -\ln\left(\frac{E(p,p')E(p,p'')\,\mathrm{d}\xi_p(p)}{E(p',p'')}\right) \\
\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)\left(\frac{\partial}{\partial t_{\tilde{p},0}} - \frac{\partial}{\partial t_{\tilde{p}',0}}\right) F_0 &= -\ln\left(\frac{E(p,\tilde{p}')E(p',\tilde{p})}{E(p,\tilde{p})E(p',\tilde{p}')}\right)
\end{aligned}$$

The second derivatives of F_0 do not depend on the form ω, and thus do not depend on the times. Thus we have:

$$F_0 = \frac{1}{2}\sum_{k,l} t_k\, t_l\, \frac{\partial^2 F_0}{\partial t_k\, \partial t_l} \tag{2.45}$$

3 Reconstruction formula

In this section, we review the reconstruction of a Lax matrix and an isospectral integrable system from the spectral curve. This so-called algebro-geometric reconstruction was first discovered by Krichever [Kri77], and we merely rephrase it. The only difference is that, instead of Baker-Akhiezer functions, we prefer to use a "Baker-Akhiezer spinor kernel", which turns out to be a more intrinsic object.

3.1 Baker-Akhiezer spinor kernel

Define the 1-form:

$$\begin{aligned}\chi(z;\vec{t}) &= \sum_{p\in\overline{\mathcal{P}}} t_{p,0}\,\mathrm{d}S_{p,o}(z) + \sum_{p\in\overline{\mathcal{P}}}\sum_{j\geq 1} t_{p,j}\,\omega_{p,j}(z) \\ &= \omega(z) - 2i\pi\sum_{i=1}^{\mathfrak{g}} \epsilon_i\,\mathrm{d}u_i(z)\end{aligned} \tag{3.1}$$

which depends linearly on the times (and not on the filling fractions). By construction χ is normalized on $\mathcal{A}$-cycles:

$$\oint_{\mathcal{A}_i} \chi = 0 \tag{3.2}$$

Then we define the vector $\zeta(\vec{t}) = [\zeta_i(\vec{t})]_{1\leq i\leq \mathfrak{g}}$ with coordinates:

$$\zeta_i(\vec{t}) = \frac{1}{2i\pi}\oint_{\mathcal{B}_i}\chi = \frac{1}{2i\pi}\left(\oint_{\mathcal{B}_i}\omega - \sum_{j=1}^{\mathfrak{g}}\tau_{i,j}\oint_{\mathcal{A}_i}\omega\right) \tag{3.3}$$

which we write for short as:

$$\zeta(\vec{t}) = \frac{1}{2i\pi}\oint_{\mathcal{B}-\tau\mathcal{A}}\omega \tag{3.4}$$

It can be decomposed as:

$$\zeta(\vec{t}) = \sum_{p\in\overline{\mathcal{P}}}\sum_{j\geq 0} t_{p,j}\mathbf{v}_{p,j} = \sum_{k=(p,j)} t_k\mathbf{v}_k, \qquad \mathbf{v}_k = \frac{1}{2i\pi}\oint_{\mathcal{B}}\omega_k \tag{3.5}$$

The vector $\zeta(\vec{t})$ is a linear function of the times t_k and is independent of the filling fractions ϵ_i. In other words, it follows a linear motion with constant velocity $\mathbf{v}_k$ in the Jacobian, as a function of any of the times. A well-known property of integrable

systems is that, in appropriate variables, the motion (with any of the time t_k) is uniform and linear. The algebraic reconstruction takes the linear evolution in the Jacobian of $\mathcal{C}$ as starting point, and produces more intricated quantities whose evolution is described by a Lax system.

Definition 3.1 *We now define the Baker-Akhiezer spinor kernel as the* $(1/2, 1/2)$ *form:*

$$\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t}) = \frac{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\vec{t}) + \mathbf{c})}{E(z_1, z_2)\;\; \theta(\zeta(\vec{t}) + \mathbf{c})}\;\; \mathrm{e}^{\int_{z_2}^{z_1} \chi(z;\vec{t})}$$

where $\mathbf{c}$ *is a non-singular, odd characteristics.*

We write a subscript $_{\mathrm{cl}}$ to distinguish the algebraic construction (associated to a "classical" or dispersionless integrable system) from the one proposed in the second part of the article. This kernel is also known as the Szegő kernel [Sze21], twisted by the exponential term $\mathrm{e}^{\int_{z_2}^{z_1} \chi(z;\vec{t})}$.

Theorem 3.1 $\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})$ *is a globally defined spinor in* $(z_1, z_2) \in \mathcal{C} \times \mathcal{C}$*, i.e. it is the squareroot of a symmetric* $(1,1)$*-form.*

- *It has a simple pole at* $z_1 = z_2$*: in any local coordinate* $\xi(z)$

$$\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t}) \underset{z_1 \to z_2}{\sim} \frac{1}{E(z_1, z_2)} \sim \frac{\sqrt{\mathrm{d}\xi(z_1)\,\mathrm{d}\xi(z_2)}}{\xi(z_1) - \xi(z_2)}$$

- *It has essential singularities when* z_1 *(resp.* z_2*) approaches a pole of* ω*.*

Proof. The behavior at $z_1 \to z_2$ is obvious, and the essential singularities at the poles of ω come from the exponential term. Next, we need to prove that $\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})$ is unchanged when z_1 (resp. z_2) goes around a non-trivial cycle. When z_1 (resp. z_2) goes around an $\mathcal{A}$-cycle, the vector $\mathbf{u}(z_1)$ (resp. $\mathbf{u}(z_2)$) is translated by an integer vector, θ is thus unchanged, and thanks to Eqn. 3.2, ψ_{cl} is unchanged when z_1 (resp. z_2) goes around an $\mathcal{A}$-cycle. When z_1 (resp. z_2) goes around a $\mathcal{B}$-cycle, the vector $\mathbf{u}(z_1)$ (resp. $\mathbf{u}(z_2)$) is translated by a lattice vector of the form $\tau \cdot \mathbf{n}$ with $\mathbf{n} \in \mathbb{Z}^{\mathfrak{g}}$, and θ gets multiplied by a phase according to Eqn. 2.13. Remember that the prime form $E(z_1, z_2)$ is also a θ function, and also gets a phase given by Eqn. 2.13. ψ_{cl} is thus changed by:

$$\psi_{\mathrm{cl}}(z_1 + \mathbf{n}\mathcal{B}, z_2\,;\,\vec{t}) \to \psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})\; \mathrm{e}^{-2i\pi\,\mathbf{n}\cdot\zeta(\vec{t})}\;\; \mathrm{e}^{\mathbf{n}\cdot\oint_{\mathcal{B}}\chi} \tag{3.6}$$

and because of Eqn. 3.3, i.e. $\zeta = \frac{1}{2i\pi}\oint_{\mathcal{B}}\chi$, we see that ψ_{cl} is unchanged when z_1 (resp. z_2) goes around a $\mathcal{B}$-cycle. $\square$

3.2 Duality equation

Then we construct the following spinor matrix of size $d \times d$:

$$\Psi_{\rm cl}(x_1, x_2\,;\,\vec{t}) = [\psi_{\rm cl}(z^i(x_1), z^j(x_2)\,;\,\vec{t})]_{i,j=1}^d \tag{3.7}$$

where we recall that $z^i(x)$ are the d preimages of x on the curve $\mathcal{C}$, i.e. $X(z^i(x)) = x$, and $d = \deg X$. These preimages are distinct and this matrix is well-defined when x_1 (or x_2) is not at a branchpoint.

Theorem 3.2 *We have the "duality" equation:*

$$\Psi_{\rm cl}(x_1, x_2\,;\,\vec{t})\,\Psi_{\rm cl}(x_2, x_3\,;\,\vec{t}) = \frac{(x_1 - x_3)\,\mathrm{d}x_2}{(x_1 - x_2)(x_2 - x_3)}\;\Psi_{\rm cl}(x_1, x_3\,;\,\vec{t})$$

Proof.

$$\frac{1}{\mathrm{d}X(z)}\,\psi_{\rm cl}(z^i(x_1), z\,;\,\vec{t})\,\psi_{\rm cl}(z, z^j(x_3)\,;\,\vec{t}) \tag{3.8}$$

is a meromorphic function of z. Indeed, the product of two (1/2)-forms is a 1-form, and when we divide by $\mathrm{d}X$, we get a function. The essential singularities coming from the exponentials cancel in the product, so this function can only have poles, i.e. it is meromorphic. The only possible poles are at $z = z^i(x_1)$ or $z = z^j(x_3)$ or at the zeroes of $\mathrm{d}X(z)$. Then, summing over all sheets, we see that

$$\sum_k \frac{\psi_{\rm cl}(z^i(x_1), z^k(x_2)\,;\,\vec{t})\,\psi_{\rm cl}(z^k(x_2), z^j(x_3)\,;\,\vec{t})}{\mathrm{d}X(z^k(x_2))} \tag{3.9}$$

is a symmetric sum of a meromorphic function over all sheets of x_2, therefore it is a meromorphic function of $x_2 \in \widehat{\mathbb{C}}$, i.e. a rational function of the complex variable x_2. It remains to find its poles. $1/\mathrm{d}X(z^k(x_2)))$ behaves like $O(x_2 - X(a_i))^{-1/2}$ at ramification points, and since a rational function of x_2 cannot have a singularity of power $-1/2$, this means that this rational function has no pole at branchpoints. Its only poles can then be at $x_2 = x_1$ or $x_2 = x_3$, and they are simple poles. The residues of the corresponding poles are easily computed and give the theorem. $\square$

Theorem 3.3 *We have a refined version of the duality equation:*

$$\psi_{\rm cl}(z_1, z\,;\,\vec{t})\,\psi_{\rm cl}(z, z_2\,;\,\vec{t}) = -\psi_{\rm cl}(z_1, z_2\,;\,\vec{t})\,\left(\mathrm{d}S_{z_1,z_2}(z) - 2i\pi\sum_{j=1}^{\mathfrak{g}} \alpha_j(z_1, z_2\,;\,\vec{t})\,\mathrm{d}u_j(z)\right)$$

where

$$\alpha_j(z_1, z_2\,;\,\vec{t}) = \frac{\theta_{u_j}(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\vec{t}) + \mathbf{c})}{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\vec{t}) + \mathbf{c})} - \frac{\theta_{u_j}(\zeta(\vec{t}) + \mathbf{c})}{\theta(\zeta(\vec{t}) + \mathbf{c})}$$

Notice that Theorem 3.2 is a corollary of Theorem 3.3. Indeed the duality equation (Theorem 3.2) can be obtained by summing the equation above on all sheets $z = z^k(x)$, because $\sum_k \mathrm{d}u_i(z^k(x)) = 0$ and

$$\sum_k \mathrm{d}S_{z_1,z_2}(z^k(x)) = \frac{(X(z_1) - X(z_2))\, \mathrm{d}X(z)}{(X(z) - X(z_1))(X(z) - X(z_2))} \tag{3.10}$$

Proof. Notice that $\psi_{\mathrm{cl}}(z_1, z\,;\,\vec{t})\psi_{\mathrm{cl}}(z, z_2\,;\,\vec{t})$ is a meromorphic 1-form in z, since it has no essential singularity. It has simple poles at $z = z_1$ and $z = z_2$, with residues $\mp\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})$, and it has no other pole. This means that $\psi_{\mathrm{cl}}(z_1, z\,;\,\vec{t})\psi_{\mathrm{cl}}(z, z_2\,;\,\vec{t}) + \psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})\, \mathrm{d}S_{z_1,z_2}(z)$ is a holomorphic 1-form, with no poles, therefore it must be a linear combination of the $\mathrm{d}u_i(z)$'s, which we choose to write:

$$\psi_{\mathrm{cl}}(z_1, z\,;\,\vec{t})\, \psi_{\mathrm{cl}}(z, z_2\,;\,\vec{t}) = -\, \psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t})\, \left(\mathrm{d}S_{z_1,z_2}(z) - 2i\pi \sum_{j=1}^{\mathfrak{g}} \alpha_j(z_1, z_2\,;\,\vec{t})\, \mathrm{d}u_j(z) \right) \tag{3.11}$$

The left hand side is a well-defined spinor of z_1 and z_2, whereas in the right hand side, $dS_{z_1,z_2}(z) = \int_{z_2}^{z_1} B(z,\cdot)$ gets some shifts when z_1 or z_2 go around non-trivial cycles. This implies the following relations for the coefficients $\alpha_j(z_1, z_2)$:

$$\alpha_j(z_1 + \mathcal{A}_k, z_2\,;\,\vec{t}) = \alpha_j(z_1, z_2\,;\,\vec{t}) \quad , \quad \alpha_j(z_1, z_2 + \mathcal{A}_k\,;\,\vec{t}) = \alpha_i(z_1, z_2\,;\,\vec{t}) \tag{3.12}$$

$$\alpha_j(z_1 + \mathcal{B}_k, z_2\,;\,\vec{t}) = \alpha_j(z_1, z_2\,;\,\vec{t}) - 2i\pi\delta_{j,k} \quad , \quad \alpha_j(z_1, z_2 + \mathcal{B}_k\,;\,\vec{t}) = \alpha_j(z_1, z_2\,;\,\vec{t}) + 2i\pi\delta_{j,k} \tag{3.13}$$

Moreover, we must have $\alpha_j(z_1, z_1\,;\,\vec{t}) = 0$, and $\alpha_j(z_1, z_2\,;\,\vec{t})$ may have poles when $\psi_{\mathrm{cl}}(z_1, z_2\,;\,\vec{t}) = 0$. Apart from those poles, $\alpha_j(z_1, z_2\,;\,\vec{t})$ has no other singularities. The following quantity has all the required properties:

$$\frac{\theta_{u_j}(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta + \mathbf{c})}{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta + \mathbf{c})} - \frac{\theta_{u_j}(\zeta + \mathbf{c})}{\theta(\zeta + \mathbf{c})} \tag{3.14}$$

So, the difference of α_j and that quantity should be a meromorphic function of z_1 and z_2 without poles, i.e. a constant, and its value is zero by looking at $z_1 = z_2$. □

3.3 Link with Baker-Akhiezer functions

Given $\mathfrak{g}$ points $p_1, \dots, p_{\mathfrak{g}} \in \mathcal{C}$ in generic position, the Baker-Akhiezer function $\psi_{\mathrm{BA}}(z)$ is usually defined as the unique (up to a multiplicative constant) function on $\mathcal{C}$, with poles at the points $p_1, \dots, p_{\mathfrak{g}}$, and essential singularities at the poles q of a meromorphic 1-form σ on $\mathcal{C}$, such that $\ln \Psi_{\mathrm{BA}}(z) = \int^z \sigma + O(1)$ when $z \to q$. It can be readily written as:

$$\psi_{\mathrm{BA}} = \exp\left(\int_o^z \sigma \right) \frac{\theta(\mathbf{u}(z) - \sum_{j=1}^{\mathfrak{g}} \mathbf{u}(p_j) - \mathbf{k} + \zeta[\sigma])}{\theta(\mathbf{u}(z) - \sum_{j=1}^{\mathfrak{g}} \mathbf{u}(p_j) - \mathbf{k})} \tag{3.15}$$

We can obtain such a function by specializing the spinor kernel associated to the 1-form σ:

$$\psi_{\mathrm{BA}}(z) = \frac{\psi(z,p_1)}{\sqrt{\mathrm{d}X(z)\mathrm{d}X(p_1)}} \tag{3.16}$$

Indeed, by Jacobi inversion theorem, we can always represent $\mathbf{u}(p_1)-\mathbf{c}-\mathbf{k}$ as $\sum_{j=1}^{\mathfrak{g}} \mathbf{u}(q_j)$ for some points $q_1,\dots,q_{\mathfrak{g}} \in \mathcal{C}$. Moreover, according to the description of the zero set of the Theta function, $q_1,\dots,q_j$ are the $\mathfrak{g}$ zeroes of $\theta(\mathbf{u}(z)-\mathbf{u}(z_1)+\mathbf{c})$, in particular $p_1 \in \{q_1,\dots,q_{\mathfrak{g}}\}$.

3.4 Eigenvectors and Baker-Akhiezer functions

The usual formulation of integrable systems is obtained by specializing one of the points to $x=\infty$. In some sense, we would like to consider:

$$\psi_{\mathrm{cl}|i}(z) \text{ "}=\text{" } \psi_{\mathrm{cl}}(z,\infty_i) \tag{3.17}$$

The problem is, that the expression in the right hand side is divergent, and thus we again need regularization.

The definitions in this paragraph also apply to the spinor kernel constructed in Section 5.3, so we drop here the $_{\mathrm{cl}}$ index. We define:

$$\psi_{i,0}(z) = \lim_{z_2\to\infty_i} \frac{\psi(z,z_2;\vec{t})}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_2)}}\,\mathrm{e}^{V_{\infty_i}(z_2)}\,\left(\xi_{\infty_i}(z_2)\right)^{t_{\infty_i,0}} \tag{3.18}$$

and if $d_{\infty_i}>1$, we define for $0\leq j\leq (d_{\infty_i}-1)$:

$$\psi_{i,j}(z) = \frac{\mathrm{d}^j}{\mathrm{d}\xi_{\infty_i}(z_2)^j}\,\left(\frac{\psi(z,z_2;\vec{t})}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_2)}}\,\mathrm{e}^{V_{\infty_i}(z_2)}\,\left(\xi_{\infty_i}(z_2)\right)^{t_{\infty_i,0}}\right)_{z_2=\infty_i} \tag{3.19}$$

There are d pairs $I=(i,j)$ such that $0\leq j\leq d_{\infty_i}-1$, and therefore the vector:

$$\vec{\psi}(z) = [\psi_I(z)] \tag{3.20}$$

is a d-dimensional vector, and the matrix:

$$\Psi(x;\vec{t}) = [\psi_I(z^k(x))]_{I,1\leq k\leq d} \tag{3.21}$$

is a $d\times d$ square matrix.

Similarly, we define the dual Baker-Akhiezer functions:

$$\phi_{i,0}(z) = \lim_{z_1\to\infty_i} \frac{\psi(z_1,z;\vec{t})}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_1)}}\,\mathrm{e}^{-V_{\infty_i}(z_1)}\,\left(\xi_{\infty_i}(z_1)\right)^{-t_{\infty_i,0}} \tag{3.22}$$

and if $d_{\infty_i} > 1$, we define for each $0 \leq j \leq (d_{\infty_i} - 1)$:

$$\phi_{i,j}(z) = \frac{\mathrm{d}^j}{\mathrm{d}\xi_{\infty_i}(z_1)^j} \left(\frac{\psi(z_1, z; \vec{t})}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_1)}} \mathrm{e}^{-V_{\infty_i}(z_1)} \left(\xi_{\infty_i}(z_1)\right)^{-t_{\infty_i,0}} \right)_{z_1=\infty_i} \tag{3.23}$$

There are d pairs $I = (i, j)$ such that $0 \leq j \leq d_i - 1$, and therefore the vector:

$$\vec{\phi}(z) = [\phi_I(z)] \tag{3.24}$$

is a d-dimensional vector, and the matrix:

$$\Phi(x; \vec{t}) = [\phi_I(z^k(x))]_{I, 1 \leq k \leq d} \tag{3.25}$$

is a $d \times d$ square matrix.

3.5 Christoffel-Darboux relations

Theorem 3.4 *The matrix*

$$A_{\mathrm{cl}}^{-1} = \frac{1}{\mathrm{d}x}\ \Phi_{\mathrm{cl}}(x)\, \Psi_{\mathrm{cl}}^t(x)$$

is invertible, and independent of x. The matrix A_{cl} is called the Christoffel-Darboux matrix. This can also be written:

$$\Psi_{\mathrm{cl}}^t(x)\, A_{\mathrm{cl}}\, \Phi_{\mathrm{cl}}(x) = \mathrm{id}\ \mathrm{d}x$$

Proof. This is a mere application of Theorem 3.2, up to a conjugation. Indeed:

$$\begin{aligned}
& (A_{\mathrm{cl}}^{-1})_{(i,k),(i',k')} \\
= \; & \frac{\mathrm{d}^{k'-1}}{\mathrm{d}\xi_{\infty_{i'}}^{k'-1}(z_1)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \Big[\\
& \sum_m \frac{\psi_{\mathrm{cl}}(z_1, z^m)\psi_{\mathrm{cl}}(z^m, z_2)}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1)\, \mathrm{d}\xi_{\infty_i}(z_2)}}\ \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)}\ \xi_{\infty_i}(z_2)^{t_{\infty_i,0}}\, \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}} \Big]_{z_1=\infty_{i'}, z_2=\infty_i} \\
= \; & \frac{\mathrm{d}^{k'-1}}{\mathrm{d}\xi_{\infty_{i'}}^{k'-1}(z_1)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \Big[\left(\frac{1}{X(z) - X(z_1)} - \frac{1}{X(z) - X(z_2)} \right) \\
& \frac{\psi_{\mathrm{cl}}(z_1, z_2)\ \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)}\ \xi_{\infty_i}(z_2)^{t_{\infty_i,0}}\, \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}}}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1)\, \mathrm{d}\xi_{\infty_i}(z_2)}} \Big]_{z_1=\infty_{i'}, z_2=\infty_i}
\end{aligned} \tag{3.26}$$

If $i \neq i'$, the following quantity

$$\frac{\psi_{\mathrm{cl}}(z_1, z_2)\ \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)}\ \xi_{\infty_i}(z_2)^{t_{\infty_i,0}}\, \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}}}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1)\, \mathrm{d}\xi_{\infty_i}(z_2)}} \tag{3.27}$$

has a well-defined limit when $z_1 \to \infty_{i'}$ and $z_2 \to \infty_i$, and the term $\frac{1}{X(z)-X(z_1)} - \frac{1}{X(z)-X(z_2)}$ behaves like:

$$\frac{1}{X(z)-X(z_1)} \underset{z_1\to\infty_{i'}}{\sim} \xi_{\infty_{i'}}(z_1)^{d_{\infty_{i'}}} \tag{3.28}$$

so we are computing the $(k'-1)$-th derivative of $O(\xi_{\infty_{i'}}(z_1)^{d_{\infty_{i'}}})$, where $k' \leq d_{\infty_{i'}}$, and therefore we get 0, i.e.:

$$(A^{-1})_{\mathrm{cl}|(i,k),(i',k')} = 0 \qquad \text{if } i \neq i' \tag{3.29}$$

If $i = i'$, we first take the limit $z_1 \to \infty_i$, and again the term with $\frac{1}{X(z)-X(z_1)}$ vanishes. Then, remember that $\psi_{\mathrm{cl}}(z_1, z_2)$ has a simple pole at $z_1 = z_2$, and thus the derivative with respect to z_1, can generate a pole of degree k' at $z_2 = \infty_i$. Therefore, we are computing the $(k-1)$-th derivative of $O(\xi_{\infty_i}(z_2)^{d_{\infty_i}-k'})$. We get zero if $k + k' \leq d_{\infty_i}$, and therefore:

$$(A^{-1})_{\mathrm{cl}|(i,k),(i,k')} = 0 \qquad \text{if } k + k' \leq d_{\infty_i} \tag{3.30}$$

If $i = i'$ and $k + k' = d_{\infty_i} + 1$, the only non-vanishing contribution is:

$$\begin{aligned}
& (A^{-1})_{\mathrm{cl}|(i,k),(i,k')} \\
= \; & \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \lim_{z_1\to\infty_i} \Big[(-1)^{k'}\,(k'-1)!\,\frac{\xi_{\infty_i}(z_2)^{d_{\infty_i}}}{(\xi_{\infty_i}(z_1)-\xi_{\infty_i}(z_2))^{k'}} \\
& \psi_{\mathrm{cl}}(z_1,z_2)\;E(z_1,z_2)\;\mathrm{e}^{V_{\infty_i}(z_2)-V_{\infty_i}(z_1)}\;\xi_{\infty_i}(z_2)^{t_{\infty_i,0}}\,\xi_{\infty_i}(z_1)^{-t_{\infty_i,0}}\Big]_{z_2=\infty_i} \\
= \; & \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \Big[(-1)^{k'}\,(k'-1)!\;\xi_{\infty_i}(z_2)^{d_{\infty_i}-k'} \\
& \lim_{z_1\to\infty_i} \psi(z_1,z_2)\;E(z_1,z_2)\;\mathrm{e}^{V_{\infty_i}(z_2)-V_{\infty_i}(z_1)}\;\xi_{\infty_i}(z_2)^{t_{\infty_i,0}}\,\xi_{\infty_i}(z_1)^{-t_{\infty_i,0}}\Big]_{z_2=\infty_i} \\
= \; & (-1)^{k'}\,(k'-1)!\,(k-1)! \\
\neq \; & 0
\end{aligned} \tag{3.31}$$

The matrix A_{cl}^{-1} has thus typically the shape:

$$A_{\mathrm{cl}}^{-1} = \begin{pmatrix} & & * & & \\ & * & * & & \\ * & * & * & & \\ & & & & * \\ & & & * & * \end{pmatrix} \tag{3.32}$$

It is made of (Inverted) triangular blocks. Since the diagonal of each triangle is non-zero, this proves that the matrix A_{cl}^{-1} is invertible.

Then, if $i = i'$ and $k + k' \geq d_{\infty_i} + 1$, we write that:

$$\frac{1}{X(z)-X(z_1)} = -\frac{1}{X(z_1)} + O(1/X(z_1)^2) \tag{3.33}$$

and we see that the leading term $\frac{1}{X(z_1)}$ is independent of $X(z)$, and the part which depends on $X(z)$ is $O(1/X(z_1)^2) = O(\xi_{\infty_i}(z_1)^{2d_{\infty_i}})$. A non vanishing contribution to the part which depends on $X(z)$ could occur only if $k+k' > 2d_{\infty_i}$, which can never happen since we assumed $k, k' \leq d_{\infty_i}$. This proves that $A_{\rm cl}$ is independent of $X(z)$. $\square$

Corollary 3.1 *The matrices $\Psi_{\rm cl}(x;\vec{t})$ and $\Phi_{\rm cl}(x;\vec{t})$ are invertible.*

As a consequence, $\psi_{\rm cl}(z_1,z_2)$ can be identified with an integrable kernel. This means that we have:

Theorem 3.5 *Christoffel-Darboux relation:*

$$\psi_{\rm cl}(z_1,z_2) = \frac{\sum_{I,J} \psi_{{\rm cl}|I}(z_1)\, A_{{\rm cl}|I,J}\, \phi_{{\rm cl}|J}(z_2)}{X(z_1)-X(z_2)}$$

Proof. This is an application of Theorem 3.4 and Theorem 3.2. Indeed, we have from the duality equation, for any x, and for any invertible matrix C:

$$\begin{aligned}
& (X(z_1)-X(z_2))\, \Psi_{\rm cl}(X(z_1),X(z_2)) \\
= \;& \Psi_{\rm cl}(X(z_1),x)\Psi_{\rm cl}(x,X(z_2))\, \frac{(X(z_1)-x)(X(z_2)-x)}{{\rm d}x} \\
= \;& \Psi_{\rm cl}(X(z_1),x)C(x)\; C^{-1}(x)\Psi_{\rm cl}(x,X(z_2))\, \frac{(X(z_1)-x)(X(z_2)-x)}{{\rm d}x}
\end{aligned} \tag{3.34}$$

The very definition of the $\psi_{{\rm cl}|I}$'s, means exactly that there exists a matrix $C_{\rm cl}(x)$ such that:

$$\Psi^t_{\rm cl}(X(z_1)) = \lim_{x\to\infty} \Psi_{\rm cl}(X(z_1),x)\, C_{\rm cl}(x) \tag{3.35}$$

and similarly, there exists a matrix $\tilde{C}_{\rm cl}$ such that:

$$\Phi_{\rm cl}(X(z_2)) = \lim_{x\to\infty} \tilde{C}_{\rm cl}(x)\;\Psi_{\rm cl}(x,X(z_2)) \tag{3.36}$$

When $z_1=z_2$, we have $\Psi_{\rm cl}(x,X(z_1))\Psi_{\rm cl}(X(z_1),x) = -\frac{{\rm d}x\,{\rm d}X(z_1)}{(x-X(z_1))^2}\,{\rm id}$, which implies:

$$-\lim_{x\to\infty} \frac{{\rm d}x\,{\rm d}X(z_1)}{(x-X(z_1))^2}\, \tilde{C}_{\rm cl}(x)\, C_{\rm cl}(x) = \Phi_{\rm cl}(X(z_1))\, \Psi^t_{\rm cl}(X(z_1)) = A_{\rm cl}^{-1}\, {\rm d}X(z_1) \tag{3.37}$$

and therefore:

$$C_{\rm cl}^{-1}(x) = A_{\rm cl}\, \tilde{C}_{\rm cl}(x)\, {\rm d}x \tag{3.38}$$

This means that:

$$\begin{aligned}
& (X(z_1)-X(z_2))\, \Psi_{\rm cl}(X(z_1),X(z_2)) \\
= \;& \lim_{x\to\infty} \Psi_{\rm cl}(X(z_1),x)C_{\rm cl}(x)\, A_{\rm cl}\, \tilde{C}_{\rm cl}(x)\Psi_{\rm cl}(x,X(z_2))\, \frac{(X(z_1)-x)(X(z_2)-x)}{{\rm d}x} \\
= \;& \Psi^t_{\rm cl}(X(z_1))\, A_{\rm cl}\, \Phi_{\rm cl}(X(z_2)).
\end{aligned} \tag{3.39}$$

$\square$

3.6 Lax matrix

As corollary of Theorem 3.2:

Corollary 3.2 *If x_1 and x_2 are not branchpoints, and if $x_2 \neq x_1$, then the matrix $\Psi_{\rm cl}(x_1,x_2)$ is invertible, and:*

$$\Psi_{\rm cl}(x_1,x_2\,;\,\vec{t})\Psi_{\rm cl}(x_2,x_1\,;\,\vec{t}) = -\,\frac{{\rm d}x_1\,{\rm d}x_2}{(x_1-x_2)^2}\,{\rm id}$$

Proof. Take $x_3 = x_1$ in the duality relation. □

Corollary 3.3 *Reconstruction formula. Let*

$$\tilde{L}(x) = {\rm diag}(Y_1(x),\dots,Y_d(x)) = {\rm diag}(Y(z^1(x)),Y(z^2(x)),\dots,Y(z^d(x)))$$

For every x_1, the matrix:

$$L_{x_1}(x;\vec{t}) = \Psi_{\rm cl}(x_1,x\,;\,\vec{t})\,\tilde{L}(x)\,\Psi_{\rm cl}^{-1}(x_1,x;\vec{t}) = -\frac{(x_1-x)^2}{{\rm d}x_1\,{\rm d}x}\,\Psi_{\rm cl}(x_1,x;\vec{t})\,\tilde{L}(x)\,\Psi_{\rm cl}(x,x_1;\vec{t})$$

is a rational function of x. Its characteristic polynomial is independent of the times $\vec{t}$, and its zero locus defines the spectral curve:

$$\det\left(y-L_{x_1}(x;\vec{t})\right) = \det(y-\tilde{L}(x)) = \mathcal{E}(x,y)$$

Changing x_1 just amounts to a conjugation:

$$L_{x_1'}(x;\vec{t}) = \Psi_{\rm cl}(x_1',x_1;\vec{t})\;\tilde{L}_{x_1}(x;\vec{t})\;\Psi_{\rm cl}^{-1}(x_1',x_1;\vec{t})$$

Theorem 3.6 *The matrix $L_{x_1}(x;\vec{t})$ obeys the Lax equation:*

$$\frac{\partial}{\partial t_k}\,L_{x_1}(x;\vec{t}) = [M_{t_k;x_1}(x;\vec{t}),L_{x_1}(x;\vec{t})]$$

with respect to any of the times t_k, and where the matrix $M_{t_k;x_1}$ is:

$$M_{t_k;x_1}(x;\vec{t}) = \frac{\partial}{\partial t_k}\Psi_{\rm cl}(x_1,x;\vec{t})\;\Psi_{\rm cl}^{-1}(x_1,x;\vec{t})$$

Therefore, from any plane curve $\mathcal{C}\,:\,\mathcal{E}(x,y)=0$, we have constructed a Lax system. Since the spectral curve here coincide with $\mathcal{C}$ and is independent of the times, the evolution is said "isospectral". This reconstruction of the Lax matrix from the spectral curve, is well-known in the theory of integrable systems [BBT02]. Usually, it is presented with special values of x_1, for instance x_1 is most often chosen as a pole of X.

3.7 Differential systems

Baker-Akhiezer functions satisfy simultaneously several first order differential systems with respect to the spectral parameter and the times.

Theorem 3.7 *The matrix $\Psi_{\text{cl}}(x_1, x; \vec{t})$ is solution of a linear ODE with respect to the spectral parameter x:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{id}}{x - x_1} - M_{x;x_1}(x\,;\,\vec{t})\right) \frac{\Psi_{\text{cl}}(x_1, x\,;\,\vec{t})}{\sqrt{\mathrm{d}x\,\mathrm{d}x_1}} = 0$$

where $M_{x;x_1}$ is a rational function of x having poles only on $\overline{\mathcal{P}}$, with the same order as those of $\omega(z)/\mathrm{d}X(z)$.

Proof. Write $x = X(z)$, we have:

$$\begin{aligned} &\mathrm{d}X(z)\ \left[M_{x;x_1}(X(z);\vec{t})\right]_{i,j} - \delta_{i,j}\frac{\mathrm{d}X(z)}{(X(z)-x_1)} \\ &= -(X(z)-x_1)^2 \sum_k \mathrm{d}\left(\frac{\psi_{\text{cl}}(z^i(x_1), z^k)}{\sqrt{\mathrm{d}X(z)\,\mathrm{d}x_1}}\right) \frac{\psi_{\text{cl}}(z^k, z^j(x_1))}{\sqrt{\mathrm{d}X(z)\,\mathrm{d}x_1}} \end{aligned} \tag{3.40}$$

In the right hand side, the essential singularities cancel, and only meromorphic singularities remain. Since we perform the sum over all sheets, the result is necessarily a rational function of $X(z)$. Poles could occur at singularities of ψ, or also at zeroes of $\sqrt{\mathrm{d}X(z)}$, or at $X(z) = x_1$.

ψ_{cl} has a simple pole at $x = x_1$, i.e. at $z^k = z^i(x_1)$ for some k. Taking the derivative yields a double pole, and multiplying by $(X(z) - x_1)^2$ cancels the double pole. If $i = j$, there is also a simple pole coming from $\psi_{\text{cl}}(z^k, z^j(x_1))$, and if $i \neq j$ there is no pole. It is easy to check that the residue of the pole at $x = x_1$ is $-\delta_{i,j}$. This implies that $M_{x;x_1}(x)$ has no pole at $x = x_1$. From its definition, $M_{x,x_1}(X(z))$ must behave as $o((X(z) - X(a))^{-1/2})$ at a ramification point a, and since it is a rational fraction of $X(z)$, this must actually be $O(1)$, meaning that $M_{x;x_1}(x)$ has no poles when x approaches a branchpoint.

Near $z \to p \in \bar{\mathcal{P}}$, the only singularity comes from the exponential term and we have

$$\begin{aligned} \left[M_{x;x_1}(X(z);\vec{t})\right]_{i,j} &= (X(z)-x_1)^2 \sum_k \omega(z^k)\ \frac{\psi_{\text{cl}}(z^i(x_1), z^k)\,\psi_{\text{cl}}(z^k, z^j(x_1))}{(\mathrm{d}X(z))^2\ \mathrm{d}x_1} \\ &\quad + O(1) \end{aligned} \tag{3.41}$$

This shows that $\mathrm{d}x\, M_{x;x_1}(x;\vec{t})$ has poles in $\bar{\mathcal{P}}$, of order at most that already present in ω/dX. □

Theorem 3.8 *We have with respect to the times $t_{p,j}$ ($j \geq 1$):*

(i) $\left(\partial_{t_{p,j}} - M_{t_{p,j};x_1}(x\,;\,\vec{t})\right)\Psi_{\rm cl}(x_1, x\,;\,\vec{t}) = 0$.

(ii) $M_{t_{p,j};x_1}$ *is a rational function of* x*, with possible poles only at* $x = X(p)$.

If $X(p) \neq \infty$*, the pole is of order* j*. If* $X(p) = \infty$*, the pole is of degree* $1 + \lfloor (j-1)/d_p \rfloor$.

One could write a similar theorem for $\partial_{t_{p,0}} - \partial_{t_{p',0}}$.

Proof. We have:

$$\left[M_{t_{p,j};x_1}(x;\vec{t})\right]_{i,j} = -\frac{(x-x_1)^2}{\mathrm{d}x\,\mathrm{d}x_1} \sum_k \partial_{t_{p,j}} \left(\psi_{\rm cl}(z^i(x_1), z^k(x))\right)\ \psi_{\rm cl}(z^k(x), z^j(x_1)) \tag{3.42}$$

and again the essential singularities cancel, and since we sum over all sheets, the result is a rational fraction of x. Since the pole of $\psi_{\rm cl}$ at $x = x_1$ has a constant residue, the derivative with respect to $t_{p,j}$ has no pole at $x = x_1$. Since we divide by $\mathrm{d}X(z)$, poles at branchpoints could occur, but $\mathrm{d}X(z)$ vanishes as $\sqrt{X(z)-a}$, and a rational fraction of x cannot have half-integer singularities, so there is no pole at branchpoints. Therefore, poles may only occur through $\partial_{t_{p,j}} \mathrm{e}^{\int_z^{z_1}\omega}$, and we find:

$$\begin{aligned}\left[M_{t_{p,j};x_1}(x;\vec{t})\right]_{i,j} &= -\frac{(x-x_1)^2}{\mathrm{d}x\,\mathrm{d}x_1} \sum_k \left(\int_{z^k(x)}^{z^i(x_1)} \omega_{p,j}\right) \psi_{\rm cl}(z^i(x_1), z^k(x))\ \psi_{\rm cl}(z^k(x), z^j(x_1)) \\ &\quad + O(1)\end{aligned} \tag{3.43}$$

which shows that $M_{t_{p,j};x_1}(x;\vec{t})$ has poles only at $x = X(p)$, and of order at most j. If $X(p) = \infty$, then $(x-x_1)^2/\mathrm{d}x$ has a pole of order $d_p - 1$, and thus $M_{t_{p,j};x_1}(x;\vec{t})$ has a pole in z of order at most $d_p + j - 1$, and thus in x it has a pole of order at most $(d_p + j - 1)/d_p$. Since $M_{t_{p,j};x_1}(x;\vec{t})$ is a rational fraction of x, the order of its pole must be integer and is thus at most $1 + \lfloor (j-1)/d_p \rfloor$. □

Again, in the usual presentation of integrable systems, the ODE's are presented after sending x_1 to ∞, i.e. in terms of $\Psi(x;\vec{t})$ and $\Phi(x;\vec{t})$.

3.8 Isomonodromic problem

We see that the matrix $\Psi_{\rm cl}(x_1, x;\vec{t})$ is a simultaneous solution of several compatible linear differential systems with rational coefficients. This leads to some isomonodromic problem.

We have

$$\frac{\partial}{\partial x}\,\Psi_{\rm cl}(x_1, x;\vec{t}) = M_{x;x_1}(x)\,\Psi_{\rm cl}(x_1, x;\vec{t}) \tag{3.44}$$

A general property of linear differential systems, is that the solution $\Psi_{\rm cl}(x_1, x;\vec{t})$ is analytical wherever $M_{x;x_1}(x)$ is analytical, and may have essential singularities at singularities of $M_{x;x_1}(x)$. The solution $\Psi_{\rm cl}(x_1, x;\vec{t})$ may have monodromies around the

singularities. Let p be one of the poles, then the monodromy after going around p is:

$$\Psi_{\rm cl}(x_1, x_+; \vec{t}) = \Psi_{\rm cl}(x_1, x_-; \vec{t})\ S_{p,x_1} \tag{3.45}$$

where x_- and x_+ mean the same value of x before and after going around the singularity. Since $M_{x;x_1}(x; \vec{t})$ is a rational function of x, it has no monodromy and we have

$$M_{x;x_1}(x_+; \vec{t}) = M_{x;x_1}(x_-; \vec{t}) \tag{3.46}$$

and it is easy to see that this implies that

$$\frac{\partial}{\partial x} S_{p,x_1} = 0 \tag{3.47}$$

In other words, the monodromy is independent of x, this is why the system is called "isomonodromic". Similarly, consider any time $t_{p,j}$, and write:

$$\frac{\partial}{\partial t_{p,j}} \Psi_{\rm cl}(x_1, x; \vec{t}) = M_{t_{p,j};x_1}(x)\, \Psi_{\rm cl}(x_1, x; \vec{t}) \tag{3.48}$$

Since $M_{t_{p,j};x_1}(x)$ is a rational function of x, it has no monodromy, and this implies that

$$\frac{\partial}{\partial t_{p,j}} S_{p,x_1} = 0 \tag{3.49}$$

In other words, the monodromy is independent of the times $t_{p,j}$. Here actually, the monodromy group has finite order since $\Psi(x_1, x; \vec{t})$ lives in a bundle over the fixed, compact algebraic curve $\mathcal{C}$). This is a very special case, which is implied by the fact that the evolution is isospectral [BBT02].

4 Dispersionless Tau function

Definition 4.1 *We shift the prepotential and define:*

$$\begin{aligned}
\tilde{F_0} &= F_0 - \sum_{i=1}^{\mathfrak{g}} \epsilon_i \frac{\partial F_0}{\partial \epsilon_i} + i\pi \sum_{j,j'=1}^{\mathfrak{g}} \epsilon_j \epsilon_{j'} \tau_{j,j'} \\
&= F_0 - \zeta \cdot \epsilon - \frac{1}{2} \epsilon \cdot \tau \cdot \epsilon \\
&= \frac{1}{2} \sum_{k,l=(p,j)} t_k t_l \frac{\partial^2 F_0}{\partial t_k t_l} + \frac{1}{2} \sum_{j,j'=1}^{\mathfrak{g}} 2i\pi \epsilon_j \epsilon_{j'} \tau_{j,j'}
\end{aligned}$$

Definition 4.2 *We define:*

$$\mathcal{T}_{\rm cl}(\mathcal{C}, \omega) = \mathrm{e}^{\tilde{F_0}}\, \theta(\zeta(\vec{t}) + \mathbf{c})$$

where $\omega = \sum_k t_k \omega_k$ is the 1-form depending linearly on the times and the filling fractions, and $\tilde{F_0}$ is the shifted prepotential associated to ω. We shall also use as notation

$$\mathcal{T}_{\rm cl}(\mathcal{C}, \omega) \equiv \mathcal{T}_{\rm cl}(\vec{t})$$

$\mathcal{T}_{\rm cl}(\vec{t})$ depends as well on the data of a non-singular odd characteristics for the Theta function. It is the Tau function associated to the solution of the problem of isospectral evolution described in Section 3.6. We call here Tau function, any function satisfying Hirota bilinear equation, and linked to Baker-Akhiezer functions by Sato relation [JMU81a, JMU81b, JMU81c]. Sections 4.1 and 4.3 prove that $\mathcal{T}_{\rm cl}$ fits in this definition. In this article, we also call $\mathcal{T}_{\rm cl}$ the "semiclassical" Tau function.

4.1 Sato relation

Sato relation [SS83] means that the Baker-Akhiezer kernel $\psi_{\rm cl}(z_1, z_2)$ is obtained from the Tau function by a Schlesinger transformation. This can be formulated as:

Theorem 4.1

$$\frac{\mathcal{T}_{\rm cl}(\omega + {\rm d}S_{z_1,z_2})}{\mathcal{T}_{\rm cl}(\omega)}\,\sqrt{{\rm d}X(z_1)\,{\rm d}X(z_2)}\,=\psi_{\rm cl}(z_1, z_2\,;\,\vec{t})$$

Proof. We add two simple poles at z_1 and z_2 by considering:

$$\omega_\lambda(z) = \omega(z) + \lambda {\rm d}S_{z_1,z_2}(z) \tag{4.1}$$

According to Theorem 2.2 we have

$$\left.\frac{\partial F_0}{\partial \lambda}\right|_{\lambda=0} = \mu_{z_1} - \mu_{z_2} = \int_{z_2}^{z_1} \omega \tag{4.2}$$

and according to Theorem 2.3 we have

$$\left.\frac{\partial^2 F_0}{\partial \lambda^2}\right|_{\lambda=0} = -\ln\left(E(z_1, z_2)^2\,{\rm d}X(z_1)\,{\rm d}X(z_2)\right) \tag{4.3}$$

Since F_0 is homogeneous of degree 2 we have

$$\begin{aligned} F_0(\omega_\lambda) &= F_0(\omega) + \lambda \left.\frac{\partial F_0}{\partial \lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2}\left.\frac{\partial^2 F_0}{\partial \lambda^2}\right|_{\lambda=0} \\ &= F_0(\omega) + \lambda \int_{z_2}^{z_1} \omega - \lambda^2 \ln\left(E(z_1, z_2)\,\sqrt{{\rm d}X(z_1)\,{\rm d}X(z_2)}\right) \end{aligned} \tag{4.4}$$

It is also easy to see that

$$\zeta \to \zeta + \frac{\lambda}{2i\pi}\oint_{\mathcal{B}-\tau\mathcal{A}} {\rm d}S_{z_1,z_2} = \zeta + \lambda(\mathbf{u}(z_1) - \mathbf{u}(z_2)) \tag{4.5}$$

Taking $\lambda = 1$ implies the theorem. $\square$

4.2 Expansion near poles

In order to recover the usual presentation of Sato relation, let $p \in \mathcal{P}$, and $U_p \subseteq \mathcal{C}$ an open neighborhood of p on which the local coordinate ξ_p is well-defined. The Schlesinger transformation:

$$\omega \to \omega + \mathrm{d}S_{z_1,z_2} \tag{4.6}$$

can also be written by Taylor expanding $\mathrm{d}S_{z_1,z_2}$ for $z_1, z_2 \in U_p$:

$$\mathrm{d}S_{z_1,z_2} = \sum_{j\geq 1} \frac{\xi_p(z_1)^j}{j}\,\omega_{p,j} - \frac{\xi_p(z_2)^j}{j}\,\omega_{p,j} \tag{4.7}$$

Thus, the times associated to p after Schlesinger transformation are:

$$\forall j \geq 1 \qquad t_{p,j} \longrightarrow t_{p,j} + \left(\frac{\xi_p(z_1)^j}{j} - \frac{\xi_p(z_2)^j}{j}\right) \tag{4.8}$$

In the literature, a notation $[z]_p$ is introduced for this special collection of time:

$$[z_1]_p = \left(\frac{\xi_p(z_1)^j}{j}\right)_{j\geq 1} \tag{4.9}$$

And since $\mathrm{d}S_{z_1,z_2}(z)$ has vanishing $\mathcal{A}$-cycle integrals, the filling fractions are not changed. For example, Theorem 4.1, together with Eqn. 3.18 gives:

$$\psi_{\mathrm{cl}|(p,0)}(z_1) = \frac{\mathcal{T}_{\mathrm{cl}}(\vec{t} + [z_1]_p)}{\mathcal{T}_{\mathrm{cl}}(\vec{t})} \tag{4.10}$$

4.3 Hirota bilinear equation

For any point $z \in \mathcal{C}$, we define the insertion operator δ_z, which acts on functions of a meromorphic 1-form on $\mathcal{C}$, by considering the one parameter deformation consisting in adding a Bergman kernel $B(z,\cdot)/\mathrm{d}X(z)$:

$$\delta_z\, f(\omega) = \mathrm{d}X(z)\,\frac{\partial}{\partial\lambda}\, f\Big(\omega + \lambda\,\frac{B(z,\cdot)}{\mathrm{d}X(z)}\Big)\Big|_{\lambda=0} \tag{4.11}$$

Locally, if z is near a pole p, we have:

$$\delta_z \equiv \sum_{j\geq 1} \xi_p(z)^{j-1}\,\mathrm{d}\xi_p(z)\;\;\partial_{t_{p,j}} \tag{4.12}$$

δ_z is obviously a derivation.

Theorem 4.2 *For any two 1-forms ω and $\hat{\omega}$ defined on the same Riemann surface $\mathcal{C}$, we have:*

$$\operatorname*{Res}_{z'\to z} \psi_{\mathrm{cl}}(z,z'\,;\,\omega)\,\psi_{\mathrm{cl}}(z',z\,;\,\hat{\omega}) = -\delta_z\,\ln\frac{\mathcal{T}_{\mathrm{cl}}(\omega)}{\mathcal{T}_{\mathrm{cl}}(\hat{\omega})}$$

Proof. $\psi_{\rm cl}(z,z'\,;\,\omega)\psi_{\rm cl}(z',z\,;\,\hat{\omega})$ has a double pole at $z=z'$, and thus evaluating the residue computes a derivative. □

As a corollary, if we choose $\hat{\omega}=\omega+{\rm d}S_{z_1,z_2}$, the expression on which we take the residue is a meromorphic form in z (it has no essential singularity). Its only other pole is located at $z=z_2$, it is simple, and by moving the integration contour, we find:

Corollary 4.1 *For any data $(\mathcal{C},\omega)$, the Baker-Akhiezer kernel is self-replicating:*

$$\delta_z\,\psi_{\rm cl}(z_1,z_2)=-\psi_{\rm cl}(z_1,z)\,\psi_{\rm cl}(z,z_2)$$

This self-replication property is the analog of the Ricatti equation in [BBT02]. Notice that it can also be obtained by a straightforward computation of $\delta_z\ln\psi_{\rm cl}(z_1,z_2)$ from Definition 3.1) and comparison with the refined duality equation (Eqn. 3.3).

Associated to any derivation ∂_t, we may define a Hirota operator D_t [Hir71] acting on two functions $f(t),g(t)$, such that:

$$(D_tf\;\cdot\;g)(t)\equiv\partial_uf(t+u)g(t-u)\Big|_{u=0}=g^2(t)\partial_t(f/g) \tag{4.13}$$

This allows us to reformulate the self-replication property as:

Theorem 4.3 *In terms of $\mathcal{T}_{\rm cl}$, for any pole p, the self-replication property is equivalent to the Hirota bilinear difference equation:*

$$D_z\,\mathcal{T}_{\rm cl}(\vec{t}+[z_1]_p-[z_2]_p)\cdot\mathcal{T}_{\rm cl}(\vec{t})=-\mathcal{T}_{\rm cl}(\vec{t}+[z_1]_p-[z]_p)\mathcal{T}_{\rm cl}(\vec{t}+[z]_p-[z_2]_p)$$

Actually, this property for $\mathcal{T}_{\rm cl}$ is equivalent to the Fay identity (see § 2.2.3) satisfied by the Theta function $\theta(\cdot|\tau)$. The Hirota equation can also be written in a more symmetric way by setting:

$$\vec{t}\longleftarrow\vec{t}-\frac{[z_1]_p}{2}+\frac{[z_2]_p}{2} \tag{4.14}$$

Namely:

$$\begin{aligned}&D_z\,\mathcal{T}_{\rm cl}\left(\vec{t}+\frac{[z_1]_p}{2}-\frac{[z_2]_p}{2}\right)\cdot\mathcal{T}_{\rm cl}\left(\vec{t}-\frac{[z_1]_p}{2}+\frac{[z_2]_p}{2}\right)\\ =\;&-\mathcal{T}_{\rm cl}\left(\vec{t}-[z]_p+\frac{[z_1]_p}{2}+\frac{[z_2]_p}{2}\right)\mathcal{T}_{\rm cl}\left(\vec{t}+[z]_p-\frac{[z_1]_p}{2}-\frac{[z_2]_p}{2}\right)\end{aligned}$$

The procedure to translate it into a set of differential equations (with respect to the times) is well-known, it is merely obtained by Taylor expansion in $\xi_p(z)$, $\xi_p(z_1)$ and $\xi_p(z_2)$ (see e.g. [JM83]). This gives an infinite set of partial differential equations involving derivatives of the Tau function with respect to times $t_{p,j}$, which are equations of a KP hierarchy. These equations are equivalent to Hirota bilinear equation.

5 Dispersive deformation

So far, the spectral curve $(\mathcal{C}, X, Y)$ was fixed once for all, and was equipped with a 1-form $\omega = \sum_k t_k \omega_k$ depending linearly on times. Now, we shall let the spectral curve itself change around $(\mathcal{C}, X, Y)$, and in particular we may vary the complex structure of the curve $\mathcal{C}$. $Y\mathrm{d}X$ shall play the role of the 1-form.

5.1 More geometry: symplectic invariants

For any spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$, a sequence of symplectic invariants $F_g(\mathcal{S})$, and of symplectic covariant forms $\omega_n^{(g)}(\mathcal{S})$ was defined in [EO07a]. Let us recall their definition and main properties.

5.1.1 Topological recursion

Let a_i be the ramification points of $\mathcal{S}$, i.e. the zeroes of $\mathrm{d}X$. We assume that the spectral curve $\mathcal{S}$ is regular, i.e. a_i are simple zeroes of $\mathrm{d}X$ and $\mathrm{d}Y(a_i) \neq 0$. Then, Y behaves like a squareroot near ramification points:

$$Y(z) = Y(a_i) + Y'(a_i)\sqrt{X(z) - X(a_i)} + O(X(z) - X(a_i)) \tag{5.1}$$

and there is a unique other point $\overline{z} \neq z$ such that $X(z) = X(\overline{z})$, at least for z in a neighborhood U_{a_i} of a_i. Then we define the recursion kernel [EO07a]:

$$K(z_0, z) = -\frac{1}{2}\frac{\int_{\overline{z}}^{z} B(z_0, \cdot)}{(Y(z) - Y(\overline{z}))\,\mathrm{d}X(z)} \tag{5.2}$$

In the variable z_0, it is a meromorphic 1-form defined globally for $z_0 \in \mathcal{C}$, and in the variable z, it is the inverse of a 1-form, only defined in $\bigcup_i U_{a_i}$.

Definition 5.1 *The "symplectic covariant forms" $\omega_n^{(g)}(z_1, \dots, z_n)$ are meromorphic $\underbrace{(1, \dots, 1)}_{n \text{ times}}$-forms defined by the following recursion:*

$$\omega_1^{(0)}(z) = Y(z)\,\mathrm{d}X(z), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

and for $J = \{z_1, \dots, z_n\}$:

$$\begin{aligned}\omega_{n+1}^{(g)}(z_0, J) \;=\; & \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z)\Big(\omega_{n+2}^{(g-1)}(z, \overline{z}, J) \\ & + \sum_{0 \le h \le g, I \subseteq J}^{\text{"stable"}} \omega_{1+|I|}^{(h)}(z, I)\,\omega_{1+n-|I|}^{(g-h)}(\overline{z}, J \setminus I)\Big)\end{aligned}$$

where "stable" means that terms containing a $\omega_1^{(0)}$ factor should be excluded of the sum.

An important property proved in [EO07a], is that:

Theorem 5.1 *For $2-2g-n<0$, $\omega_n^{(g)}(z_1,\dots,z_n)$ is symmetric in its n variables, and it is a meromorphic form in each variable, having poles only at ramification points, with vanishing residues. The poles are of order at most $6g+2n-4$.*

For instance, for $(g,n)=(0,3)$, this definition amounts to:

$$\begin{aligned}\omega_3^{(0)}(z_0,z_1,z_2) &= \sum_i \operatorname*{Res}_{z\to a_i}\Big(-\frac{1}{2}\frac{\int_{\overline{z}}^{z} B(z_0,\cdot)}{(Y(z)-Y(\overline{z}))\mathrm{d}X(z)}\,(B(z,z_1)B(\overline{z},z_2)+B(\overline{z},z_1)B(z,z_2)\Big) \\ &= \sum_i \operatorname*{Res}_{z\to a_i}\frac{B(z_0,z)B(z_1,z)B(z_2,z)}{\mathrm{d}X(z)\,\mathrm{d}Y(z)} \qquad (5.3)\end{aligned}$$

Definition 5.2 *The "symplectic invariants" F_g are numbers associated to $\mathcal{S}$, as follows.*

- *For $g=0$, we define $F_0(\mathcal{S})$ as the prepotential, see Definition 2.6.*
- *For $g=1$, F_1 is defined in terms of the Bergman Tau function $\mathcal{T}_{B,X}$ introduced by Kokotov and Korotkin [KK03]:*

$$F_1=-\frac{1}{2}\ln(\mathcal{T}_{B,X})-\frac{1}{24}\ln\left(\prod_i Y'(a_i)\right)$$

  where $Y'(a_i)=\lim_{z\to a_i}(Y(z)-Y(a_i))/\sqrt{X(z)-X(a_i)}$. F_1 is related to the logarithm of the determinant of the Laplacian on $\mathcal{C}$ with metrics $|y\mathrm{d}x|^2$.

- *For $g\geq 2$:*

$$F_g(\mathcal{S})=\frac{1}{2-2g}\sum_i \operatorname*{Res}_{z\to a_i}\Phi(z)\,\omega_1^{(g)}(z)$$

  where $\Phi(z)$ is any primitive of $Y\mathrm{d}X$, i.e. $\mathrm{d}\Phi=Y\mathrm{d}X$.

As a notation we write $\omega_0^{(g)}\equiv F_g$.

The name "symplectic invariants" came from the important property, proved in [EO07b], that F_g and the cohomology class of $\omega_n^{(g)}$ are invariant under the following transformations of spectral curves, each of which let the symbolic symplectic form $\mathrm{d}X\wedge\mathrm{d}Y$ invariant:

$$(X,Y)\to(-Y,X),\qquad (X,Y)\to(X,Y+\mathrm{R}(X)),\qquad (X,Y)\to(\lambda X,Y/\lambda)\qquad (5.4)$$

where R is any rational function and $\lambda\in\mathbb{C}$.

5.1.2 Special geometry

Let us consider an infinitesimal deformation of the spectral curve $\mathcal{S}$. It implies an infinitesimal deformation of the complex structure of the Riemann surface $\mathcal{C}$, and a deformation of the form $Y\mathrm{d}X$:

$$Y\mathrm{d}X \longrightarrow Y\mathrm{d}X + \lambda\,\Omega + O(\lambda^2) \tag{5.5}$$

where Ω is a meromorphic form. We can decompose[5] as in Theorem 2.1

$$\Omega = \sum_k t_k\,\omega_k + 2i\pi \sum_{j=1}^{\mathfrak{g}} \epsilon_j\,\mathrm{d}u_j \tag{5.6}$$

There is a dual cycle associated to Ω:

$$\Omega^* = \sum_k t_k\omega_k^* + \sum_{j=1}^{\mathfrak{g}} \epsilon_j \mathcal{B}_j \tag{5.7}$$

such that:

$$\Omega(z) = \int_{\Omega^*} B(\cdot, z) \tag{5.8}$$

Theorem 5.2

$$\forall n \geq 0\,, \qquad \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} \omega_n^{(g)}(z_1,\dots,z_n) = \int_{\Omega^*} \omega_{n+1}^{(g)}(\cdot, z_1,\dots,z_n)$$

and the derivative is evaluated at $X(z_i)$ fixed.

This theorem was proved in [EO07a]. This set of relations is usually called "special geometry. Let us give some examples:

- 1st kind deformations/addition of holomorphic forms:

$$\frac{\partial\omega_n^{(g)}(z_1,\dots,z_n)}{\partial\epsilon_i} = \oint_{\mathcal{B}_i} \omega_{n+1}^{(g)}(\cdot, z_1,\dots,z_n) \tag{5.9}$$

- second kind deformations/addition of a double pole:

$$\frac{\partial\omega_n^{(g)}(z_1,\dots,z_n)}{\partial t_{p,1}} = \frac{\omega_{n+1}^{(g)}(z, z_1,\dots,z_n)}{\mathrm{d}\xi_p(z)} \tag{5.10}$$

If p is a pole of $Y\mathrm{d}X$, this does not apply to $(g,n) = (0,0)$.

[5] ω_k here belongs to the basis of meromorphic forms defined in Section 2.3.4. It has nothing to do with the invariants $\omega_n^{(g)}$.

- third kind deformations/addition of simple poles:

$$\frac{\partial \omega_n^{(g)}(z_1,\dots,z_n)}{\partial t_{p,0}} - \frac{\partial \omega_n^{(g)}(z_1,\dots,z_n)}{\partial t_{p',0}} = \int_{p'}^{p} \omega_{n+1}^{(g)}(\cdot, z_1,\dots,z_n) \tag{5.11}$$

- $(g,n) = (0,0)$. We recover the constitutive relations for the prepotential (Section 2.4):

$$\frac{\partial F_0}{\partial t_k} = \oint_{\omega_k^*} Y \mathrm{d}X \tag{5.12}$$

- $(g,n) = (0,1)$. We recover the definition of the form-cycle duality pairing by the Bergman kernel:

$$\frac{\partial Y \mathrm{d}X}{\partial t_k}(z) = \omega_k(z) = \oint_{\omega_k^*} B(\cdot, z) \tag{5.13}$$

5.2 Candidate for a dispersive Tau function

Let us consider:

$$Z_{[\mu]}[\mathcal{S},\mathbf{n}] = \exp\left(\sum_{g=0}^{\infty} N^{2-2g}\, F_g(\mathcal{S},\mathbf{n}+\mu)\right) \tag{5.14}$$

where $F_g(\mathcal{S},\mathbf{n}')$ are the symplectic invariants associated to the curve $\mathcal{S}$ and the vector of filling fractions $\mathbf{n}' = (n'_1,\dots,n'_{\mathfrak{g}})$. $\ln Z[\mathcal{S},\mathbf{n}]$ is a formal Laurent series in N (in particular we emphasize that it contains no oscillatory terms, by opposition with the definition to come). Then, we would like to define the sum over all integer filling fractions:

$$\mathcal{T}_{[\mu,\nu]}(\mathcal{S}) = \sum_{\mathbf{n}\in\mathbb{Z}^{\mathfrak{g}}} e^{2i\pi\mathbf{n}\cdot\nu}\, Z_{[\mu]}[\mathcal{S},\mathbf{n}] \tag{5.15}$$

The intuition behind this definition is Whitham averaging. The filling fractions are considered as fast evolving moduli in the system, so that we do not see a fixed spectral curve, but rather a statistical mixture of spectral curves with different filling fractions. When $\mathfrak{g} > 0$, $\ln \mathcal{T}_{[\mu,\nu]}(\mathcal{S})$ is no more a Laurent series in N. To give a precise meaning to this expression, one has to collect the terms order by order in $1/N$.

Initially, such an object has been introduced with heuristic arguments to compute the large N asymptotic of hermitian matrix integrals, and it leads to theta functions to the leading order [BDE00], and a whole series of corrections involving derivatives of theta functions [Eyn09]. Taking this series as a definition of an object $\mathcal{T}_{[\mu,\nu]}(\mathcal{S})$, some of its properties related to modularity and holomorphic anomaly equations were studied in [EM08]. In this article, we argue that $\mathcal{T}_{[\mu,\nu]}(\mathcal{S})$ is a good candidate for a Tau function.

5.2.1 Preliminaries

We shall consider the dependence of the F_g's in the filling fractions[6]:

$$F_g \equiv F_g(\epsilon_1, \ldots, \epsilon_{\mathfrak{g}})$$

and we shall write for short F_g' for the vector of first derivatives of F_g with respect to ϵ_i's:

$$F_g' = \left(\frac{\partial F_g}{\partial \epsilon_1}, \ldots, \frac{\partial F_g}{\partial \epsilon_{\mathfrak{g}}} \right),$$

and $F_g'', F_g''', \ldots, F_g^{(k)}$, denote likewise the tensor of higher derivatives.

We need Theta functions, and we shall introduce an appropriate notation for our purposes. We define the $\Theta_{[\mu,\nu]}$ function as:

$$\Theta_{[\mu,\nu]}(\mathbf{w}|\tau) = \sum_{\mathbf{p}\in\mathbb{Z}^{\mathfrak{g}}} e^{i\pi(\mathbf{p}+\mu-N\epsilon)\cdot\tau\cdot(\mathbf{p}+\mu-N\epsilon)+(\mathbf{p}+\mu-N\epsilon)\cdot\mathbf{w}+2i\pi\mathbf{p}\cdot\nu} \tag{5.16}$$

$[\mu, \nu]$ play the role of a characteristics, although we do not require it to be half-integer here. We can compare to the definition of the Siegel theta function with characteristics $[\mu, \nu]$:

$$\vartheta[^{\mu}_{\nu}](\mathbf{w}|\tau) = \sum_{\mathbf{p}\in\mathbb{Z}^{\mathfrak{g}}} \exp[i\pi(\mathbf{p}+\mu)\tau(\mathbf{p}+\mu) + 2i\pi(\mathbf{p}+\mu)(\mathbf{w}+\nu)] \tag{5.17}$$

and thus:

$$\Theta_{[\mu,\nu]}(\mathbf{w}|\tau) = e^{i\pi N^2\epsilon\cdot\tau\cdot\epsilon - N\epsilon\cdot\mathbf{w} - 2i\pi\mu\cdot\nu}\, \vartheta[^{\mu}_{\nu}](\mathbf{w}/2i\pi - N\tau\cdot\epsilon|\tau) \tag{5.18}$$

We shall omit to write the dependance in $[\mu, \nu]$, and we use the notation $\Theta', \Theta'', \ldots, \Theta^{(k)}$ for the tensor of derivatives with respect to $\mathbf{w}$.

$$\Theta' = \left(\frac{\partial \Theta}{\partial w_1}, \ldots, \frac{\partial \Theta}{\partial w_{\mathfrak{g}}} \right)^t \quad \text{and so on} \tag{5.19}$$

This $\Theta_{[\mu,\nu]}$ function satisfies the heat equation:

$$\frac{1}{2}\left(\partial_{\tau_{i,j}} + \partial_{\tau_{j,i}}\right)\, \Theta_{[\mu,\nu]}(\mathbf{w}) = i\pi\Theta''(\mathbf{w})_{i,j} \tag{5.20}$$

Again, in this equation, $\tau_{i,j}$ and $\tau_{j,i}$ are considered independent.

From now on, the dependence in the characteristics will be implicit. Given a spectral curve $(\mathcal{C}, X, Y)$ of genus $\mathfrak{g}$, and the choice of a symplectic basis $(\mathcal{A}, \mathcal{B})$, we set:

$$\begin{aligned}
\epsilon &= \frac{1}{2i\pi} \oint_{\mathcal{A}} Y \mathrm{d}X \\
\mathbf{w}_0 &= N F_0' = N \oint_{\mathcal{B}} Y \mathrm{d}X \\
\tau &= \frac{1}{2i\pi} F_0'' = \frac{1}{2i\pi} \oint_{\mathcal{B}} \oint_{\mathcal{B}} B \quad \text{the Riemann matrix of periods of } \mathcal{C}
\end{aligned}$$

[6] We warn the reader not to confuse the the genus $\mathfrak{g}$ of the spectral curve, and the index g of the symplectic invariant F_g.

and we define to shorten formulas:

$$\Theta^{(k)} \equiv \Theta^{(k)}_{[\mu,\nu]}\left(\mathbf{w} = \mathbf{w}_0 \,|\, \tau\right) \tag{5.21}$$

as the tensor of k^{th} derivatives of Θ. In the definition of Θ, notice that $\mathbf{w}_0$ appears through the combination:

$$\zeta = \mathbf{w}_0 - N\tau \cdot 2i\pi\,\epsilon = N \oint_{\mathcal{B}-\tau\mathcal{A}} Y \mathrm{d}X \tag{5.22}$$

which is modular of weight 1, and already appeared in Section 3.1.

5.2.2 Definition and comments

Definition 5.3 *For any $(\mu,\nu) \in \mathbb{C}^{\mathfrak{g}}$ and any spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$ of genus $\mathfrak{g}$, we define:*

$$\begin{aligned}
\mathcal{T}(\mathcal{S}) &= \mathrm{e}^{\sum_{g\geq 0} N^{2-2g} F_g} \sum_k \sum_{l_i>0} \sum_{h_i > 1-\frac{l_i}{2}} \frac{N^{\sum_i (2-2h_i-l_i)}}{k! l_1! \ldots l_k!}\; F^{(l_1)}_{h_1} \ldots F^{(l_k)}_{h_k}\; \Theta^{(\sum_i l_i)} \\
&= \mathrm{e}^{N^2 F_0}\,\mathrm{e}^{F_1} \Big\{ \Theta + \frac{1}{N}\Big(\Theta' F_1' + \frac{1}{6}\Theta'''\,F_0'''\Big) \\
&\quad + \frac{1}{N^2}\Big(\Theta\,F_2 + \frac{1}{2}\Theta'' F_1'' + \frac{1}{2}\Theta'' F_1'^2 + \frac{1}{24}\Theta^{(4)} F_0'''' + \frac{1}{6}\Theta^{(4)} F_0''' F_1' + \frac{1}{72}\Theta^{(6)} F_0'''^2\Big) \\
&\quad + o(1/N^2)\Big\}
\end{aligned} \tag{5.23}$$

This function is defined formally order by order in N (in the coefficient of $1/N^k$, each term $\Theta^{(j)}$ is considered formally of order 1). It can be seen as a genuine asymptotic series when the reference filling fractions ϵ are chosen such that $\mathbf{w}_0 = NF_0'$ is of order $O(1)$.

When $\mu + \tau\nu$ is a non-singular half-integer odd characteristics, the leading term when $N \to \infty$ of $\mathcal{T}(\mathcal{S})$ coincides with the classical Tau function of Section 4 computed for the differential form $\omega = N\,Y\mathrm{d}X$, up to an exponential factor:

$$\begin{aligned}
\mathcal{T}_{[\mu,\nu]}(\mathcal{S}) &\sim e^{N^2 F_0}\,\Theta(\mathbf{w}_0|\tau) \\
&\sim e^{N^2 \tilde{F}_0}\,\theta(\zeta + \nu + \mu\cdot\tau)\,e^{2i\pi\mu\cdot\zeta} \\
&\sim \mathcal{T}_{\mathrm{cl}}(NY\mathrm{d}X)\,e^{2i\pi\mu\cdot\zeta}
\end{aligned}$$

Notice that the Fay identity for θ [Fay70] presented in § 2.2.3 is also true if we multiply θ by an exponential factor of its argument. Therefore, Theorem 4.3 ensures that the large N limit of $\mathcal{T}(\mathcal{S})$ satisfies the Hirota bilinear equation (Theorem. 4.3), i.e. is a dispersionless Tau function to leading order. This is also true for arbitrary characteristics $[\mu,\nu]$, although the times have to be shifted by a (maybe complex) constant in

this case. We conjecture (see Section 7) that $\mathcal{T}_{[\mu,\nu]}$ is actually a Tau function, satisfying Hirota equations to all orders.

We also mention that under a modular transformation (i.e. a change of choice for the $\mathcal{A}$ and $\mathcal{B}$ cycles), $\mathcal{T}_{[\mu,\nu]}$ changes like the Theta function of characteristics $[\mu,\nu]$ (see [EM08]):

Proposition 5.1 *Under a modular $Sp(2\mathfrak{g},\mathbb{Z})$ transformation $\tau\to\tilde{\tau}=(A\tau+B)(C\tau+D)^{-1}$, the characteristics $[\mu,\nu]$ changes as $\mu\to\tilde{\mu}=D\mu-C\nu+\frac{1}{2}(CD^t)_{\rm diag}$, $\nu\to\tilde{\nu}=-B\mu+A\nu+\frac{1}{2}(AB^t)_{\rm diag}$, the Tau function transforms as*

$$\mathcal{T}_{[\mu,\nu]}\to\zeta_{[\mu,\nu]}(A,B,C,D)\ \mathcal{T}_{[\tilde{\mu},\tilde{\nu}]} \tag{5.24}$$

where $\zeta_{[\mu,\nu]}(A,B,C,D)$ is the phase factor, independent of the spectral curve, which already appears in the modular transformation of the Theta function.

5.3 Baker-Akhiezer spinor kernel

We now define the spinor kernel $\psi(z_1,z_2\,;\mathcal{S})$ through Sato's relation:

Definition 5.4

$$\begin{aligned}\psi(z_1,z_2\,;\mathcal{S}) &= \frac{\mathcal{T}(\mathcal{S},Y\mathrm{d}X\to Y\mathrm{d}X+\frac{1}{N}\,\mathrm{d}S_{z_1,z_2})}{\mathcal{T}(\mathcal{S})}\,\sqrt{\mathrm{d}X(z_1)\,\mathrm{d}X(z_2)}\\ &= \frac{\mathcal{T}(\mathcal{S}+[z_1]-[z_2])}{\mathcal{T}(\mathcal{S})}\,\sqrt{\mathrm{d}X(z_1)\,\mathrm{d}X(z_2)}\end{aligned}$$

where $[z]$ is Sato's notation, see Eqn. 4.9 in Section 4.1.

$\psi(z_1,z_2)$ is again defined formally, order by order in $1/N$, or as an asymptotic series for appropriate ϵ. The leading order coincides with $\psi_{\rm cl}$ introduced in Definition 3.1. Let us give the first few orders:

$$\begin{aligned}\psi(z_1,z_2\,;\mathcal{S}) &= \frac{e^{N\int_{z_2}^{z_1}Y\mathrm{d}X}}{E(z_1,z_2)}\,\frac{\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}{\Theta(\mathbf{w}_0\,|\,\tau)}\\ &\quad\Big\{1+\frac{1}{N}\Big[\frac{1}{6}\int_{z_2}^{z_1}\int_{z_2}^{z_1}\int_{z_2}^{z_1}\omega_3^{(0)}+\int_{z_2}^{z_1}\omega_1^{(1)}\\ &\quad+\frac{1}{2}\frac{\Theta'(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,\big|\,\tau)}{\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}\int_{z_2}^{z_1}\int_{z_2}^{z_1}\oint_{\mathcal{B}}\omega_3^{(0)}\\ &\quad+\frac{1}{2}\frac{\Theta''(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}{\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}\int_{z_2}^{z_1}\oint_{\mathcal{B}}\oint_{\mathcal{B}}\omega_3^{(0)}\\ &\quad+\left(\frac{\Theta'(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}{\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}-\frac{\Theta'(\mathbf{w}_0\,|\,\tau)}{\Theta(\mathbf{w}_0\,|\,\tau)}\right)F_1'\end{aligned}$$

$$+\frac{1}{6}\left(\frac{\Theta'''(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}{\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau)}-\frac{\Theta'''(\mathbf{w}_0\,|\,\tau)}{\Theta(\mathbf{w}_0\,\Big|\,\tau)}\right)F_0'''\Big]+o(1/N)\Big\}\tag{5.25}$$

Lemma 5.1 *$\psi(z_1,z_2\,;\mathcal{S})$ is a well-defined spinor in z_1 and z_2. Furthermore:*

- *$\psi(z_1,z_2)$ has a simple pole at $z_1=z_2$:*

$$\psi(z_1,z_2\,;\mathcal{S})\underset{z_1\to z_2}{\sim}\frac{\sqrt{\mathrm{d}X(z_1)\,\mathrm{d}X(z_2)}}{X(z_1)-X(z_2)}$$

- *It has an essential singularity near any pole p of $Y\mathrm{d}X$, of the form $\mathrm{e}^{N\int_{z_2}^{z_1}Y\mathrm{d}X}$.*

- *At all orders in $1/N$ (except at leading order), $\psi(z_1,z_2)$ has poles at the ramification points a_i. Their order increase with the order of $1/N$.*

Proof. When z_1 goes around an $\mathcal{A}$-cycle, $\mathrm{d}S_{z_1,z_2}$ is unchanged. When z_1 goes around a cycle $\mathcal{B}_j$, $\mathrm{d}S_{z_1,z_2}$ is shifted by a holomorphic form $\mathrm{d}S_{z_1,z_2}\to\mathrm{d}S_{z_1,z_2}+2i\pi\mathrm{d}u_j$, which is dual to a $\partial/\partial\epsilon_j$, and since the Tau function is background independent (it was proved in [EM08] that $\partial\mathcal{T}/\partial\epsilon_i=0$), then it is unchanged. This shows that $\psi(z_1,z_2\,;\mathcal{S})$ is a well-defined spinor.

Then we compute each term of $\mathcal{T}(\mathcal{S}+[z_1]-[z_2])$ by writing the Taylor expansion:

$$F_g\left(\mathcal{S},y\mathrm{d}x\to y\mathrm{d}x+\frac{\lambda}{N}\mathrm{d}S_{z_1,z_2}\right)=\sum_{n\geq 0}\frac{\lambda^n}{n!\,N^n}\left.\frac{\partial^n F_g}{\partial\lambda^n}\right|_{\lambda=0}\tag{5.26}$$

which we need to evaluate at $\lambda=1$. The n-th derivatives of F_g at $\lambda=0$ are computed by the special geometry relations theorem 5.2, using the dual cycle $(\mathrm{d}S_{z_1,z_2})^*=[z_2,z_1]$:

$$\left.\frac{\partial^n F_g}{\partial\lambda^n}\right|_{\lambda=0}=\int_{z_2}^{z_1}\dots\int_{z_2}^{z_1}\omega_n^{(g)}\tag{5.27}$$

All the $\omega_n^{(g)}$ with $2-2g-n<0$ are meromorphic, and have poles only at ramification points, without residues. This implies that their contribution to $\psi(z_1,z_2)$ provides only poles at ramification points. The only terms involving $\omega_n^{(g)}$ with $2-2g-n\geq 0$, are $\partial_\lambda F_0$, $\partial_\lambda^2 F_0$ and $\partial_\lambda F_0'$.

- $\left.\frac{\partial F_0}{\partial\lambda}\right|_{\lambda=0}=\int_{z_2}^{z_1}Y\mathrm{d}X$, which contributes to ψ as the essential singularity $\mathrm{e}^{N\int_{z_2}^{z_1}Y\mathrm{d}X}$.

- $\left.\frac{\partial^2 F_0}{\partial\lambda^2}\right|_{\lambda=0}=-\ln\left(E(z_1,z_2)^2\,\mathrm{d}X(z_1)\,\mathrm{d}X(z_2)\right)$ which contributes to ψ as $1/E(z_1,z_2)$.

- $\left.\frac{\partial F_0'}{\partial\lambda}\right|_{\lambda=0}=2i\pi\left(\mathbf{u}(z_1)-\mathbf{u}(z_2)\right)$ which does not yield any singularity.

All the other terms have $2-2g-n<0$, and contribute order by order, only as combinations of meromorphic forms and derivatives of Theta functions, having poles at ramification points conveyed by the $\omega_n^{(g)}$'s with $2-2g-n<0$ $\square$

Notice that as a corollary of Proposition 5.1, ψ has nice modular properties:

Corollary 5.1 *Under a modular $Sp(2\mathfrak{g},\mathbb{Z})$ transformation $\tau \to \tilde{\tau} = (A\tau + B)(C\tau + D)^{-1}$, the characteristics $[\mu,\nu]$ changes as $\mu \to \tilde{\mu} = D\mu - C\nu + \frac{1}{2}(CD^t)_{\rm diag}$, $\nu \to \tilde{\nu} = -B\mu + A\nu + \frac{1}{2}(AB^t)_{\rm diag}$, the spinor kernel $\psi_{[\mu,\nu]}(z_1,z_2)$ transforms as*

$$\psi_{[\mu,\nu]}(z_1,z_2) \to \ \psi_{[\tilde{\mu},\tilde{\nu}]}(z_1,z_2) \tag{5.28}$$

6 Correlators

6.1 Second kind deformations of $\mathcal{S}$

Let us recall the definition of the insertion operator δ_z, already encountered in Section 4.3 and adapted now for varying spectral curves.

Definition 6.1 *We define the insertion operator δ_z, acting on a functional $f(\mathcal{S})$ of a spectral curve $\mathcal{S}$, as follows:*

$$\delta_z f = {\rm d}X(z)\ \frac{\partial}{\partial\lambda}\ f(\mathcal{S}_\lambda)\Big|_{\lambda=0}$$

where $\mathcal{S}_\lambda$ is the family of spectral curves obtained from $\mathcal{S}$ by:

$$(Y{\rm d}X)_\lambda = Y{\rm d}X + \lambda\,\frac{B(z,\cdot)}{{\rm d}X(z)}$$

δ_z is a derivation.

The dual cycle of $B(z,\cdot)/{\rm d}X(z)$ is the contour surrounding z with index 1:

$$B(z,\cdot) = \mathop{\rm Res}_{z'\to z}\ B(z',\cdot)\ \frac{{\rm d}X(z)}{(X(z')-X(z))} \tag{6.1}$$

Then, the relations of special geometry (theorem 5.2) for $\omega_n^{(g)}$ implies, for any $n,g\geq 0$:

$$\begin{aligned}\delta_z\omega_n^{(g)}(z_1,\dots,z_n) &= \mathop{\rm Res}_{z'\to z}\ \omega_{n+1}^{(g)}(z',z_1,\dots,z_n)\ \frac{{\rm d}X(z)}{(X(z')-X(z))} \\ &= \omega_{n+1}^{(g)}(z,z_1,\dots,z_n)\end{aligned} \tag{6.2}$$

For instance:

$$\delta_z F_0 = \omega_1^{(0)}(z) = Y(z)\,{\rm d}X(z)\,, \qquad \delta_z F_g = \omega_1^{(g)}(z)\,, \qquad \delta_z\omega_1^{(0)}(z') = B(z,z') \tag{6.3}$$

6.2 Definition

Definition 6.2 *For n positive integer, we define the correlators $W_n(z_1,\dots,z_n)$ and the disconnected correlators $\overline{W}_n(z_1,\dots,z_n)$ as:*

$$\begin{aligned}
W_n(z_1,\dots,z_n) &= N^{-n}\,\delta_{z_1}\cdots\delta_{z_n}\,\ln\mathcal{T}(\mathcal{S}) \\
\overline{W}_n(z_1,\dots,z_n) &= \frac{N^{-n}\,\delta_{z_1}\cdots\delta_{z_n}\,\mathcal{T}(\mathcal{S})}{\mathcal{T}(\mathcal{S})}
\end{aligned}$$

$W_n(z_1,\dots,z_n)$ and $\overline{W}_n(z_1,\dots,z_n)$ are $\underbrace{(1,\dots,1)}_{n\text{ times}}$-forms, symmetric in their n variables. Each coefficient, order by order in $1/N$, is a meromorphic form with poles at ramification points

6.3 Examples

For instance the three first orders of W_1 are:

$$\begin{aligned}
W_1(z) = &\; N\,Y(z)\,\mathrm{d}X(z) + (\ln\Theta)'\cdot 2i\pi\mathrm{d}\mathbf{u}(z) \\
&+\frac{1}{N}\left\{\omega_1^{(1)}(z) + \frac{\Theta''}{\Theta}\cdot(i\pi\delta_z\tau) + \left[F_1'(\frac{\Theta''}{\Theta}-\frac{\Theta'^2}{\Theta^2}) + \frac{F_0'''}{6}(\frac{\Theta''''}{\Theta}-\frac{\Theta'''\Theta'}{\Theta^2})\right]\cdot 2i\pi\mathrm{d}\mathbf{u}(z)\right\} \\
&+o(1/N)
\end{aligned} \tag{6.4}$$

where we recall that:

$$\begin{aligned}
\delta_z\tau_{j,k} &= \frac{\delta_z F_0''}{2i\pi} = \frac{1}{2i\pi}\oint_{\mathcal{B}_j}\oint_{\mathcal{B}_k}\omega_3^{(0)}(\cdot,\cdot,z) \\
&= 4i\pi\sum_l \mathop{\mathrm{Res}}_{z'\to a_l} K(z,z')\,\mathrm{d}u_j(z')\mathrm{d}u_k(\overline{z'}) \\
&= 2i\pi\sum_l \mathop{\mathrm{Res}}_{z'\to a_l}\frac{B(z,z')\,\mathrm{d}u_j(z')\mathrm{d}u_k(z')}{\mathrm{d}X(z')\,\mathrm{d}Y(z')} && (6.5) \\
\omega_1^{(1)}(z) &= \sum_l \mathop{\mathrm{Res}}_{z'\to a_l} K(z,z')\,B(z',\overline{z'}) && (6.6)
\end{aligned}$$

For the 2-point correlator, the three first orders are:

$$\begin{aligned}
W_2(z_1,z_2) = &\; B(z_1,z_2) + (\ln\Theta)''\cdot 2i\pi\,\mathrm{d}\mathbf{u}(z_1)\otimes 2i\pi\,\mathrm{d}\mathbf{u}(z_2) \\
&+\frac{1}{N}\left\{\frac{\Theta'}{\Theta}\cdot\int_{\mathcal{B}}\omega_3^{(0)}(\cdot,z_1,z_2) + \left(\frac{\Theta''}{\Theta}\right)'\cdot[i\pi(\delta_{z_1}\tau)\otimes 2i\pi\mathrm{d}\mathbf{u}(z_2) + 2i\pi\mathrm{d}\mathbf{u}(z_1)\otimes i\pi(\delta_{z_2}\tau)]\right. \\
&\left.+\left[(F_1)'\left(\frac{\Theta''}{\Theta}-\frac{\Theta'^2}{\Theta^2}\right)' + \frac{F_0'''}{6}\left(\frac{\Theta''''}{\Theta}-\frac{\Theta'''\Theta'}{\Theta^2}\right)'\right]\cdot 2i\pi\,\mathrm{d}\mathbf{u}(z_1)\otimes 2i\pi\,\mathrm{d}\mathbf{u}(z_2)\right\} \\
&+o(1/N)
\end{aligned} \tag{6.7}$$

For $n \geq 3$, the leading order of the n-point correlator is a $O(1)$, and is obtained by successive applications of δ_z to the $(\ln\Theta)''$ term.

$$\begin{aligned} W_n(z_1,\cdots,z_n) \;=\;& (\ln\Theta)^{(n)} \cdot \bigotimes_{j=1}^{n} 2i\pi\, \mathrm{d}\mathbf{u}(z_j) \\ &+\frac{1}{N}\Big\{ \sum_{j=1}^{n} \Big(\frac{\Theta''}{\Theta}\Big)^{(n-1)} \cdot (i\pi\delta_{z_j}\tau) \otimes \bigotimes_{k\neq j} 2i\pi\, \mathrm{d}\mathbf{u}(z_k) \\ &+ \sum_{1\leq j<k\leq n} (\ln\Theta)^{(n-1)} \cdot \Big(\int_{\mathcal{B}} \omega_3^{(0)}(\cdot, z_j, z_k)\Big) \otimes \bigotimes_{l\neq j,k} 2i\pi\, \mathrm{d}\mathbf{u}(z_l) \\ &+\Big[(F_1)'\Big(\frac{\Theta''}{\Theta} - \frac{\Theta'^2}{\Theta^2}\Big)^{(n-1)} + \frac{F_0'''}{6}\Big(\frac{\Theta''''}{\Theta} - \frac{\Theta'''\Theta'}{\Theta^2}\Big)^{(n-1)}\Big] \cdot \bigotimes_{j=1}^{n} 2i\pi\, \mathrm{d}\mathbf{u}(z_j)\Big\} \\ &+o(1/N) \end{aligned} \tag{6.8}$$

6.4 Loop equations

Theorem 6.1 *The dispersive Tau function obeys the loop equations. Namely, let us denote by Γ a contour separating the poles of $Y\,\mathrm{d}X$, from the set of preimage of a point $x \in \mathbb{C}$.*

$$(i) \qquad \oint_{z\in\Gamma} \frac{\delta_z \mathcal{T}[\mathcal{S}] - N\,(Y\mathrm{d}X)(z)}{X(z)-x} = 0$$

$$(ii) \qquad \oint_{z\in\Gamma} \operatorname*{Res}_{z'\to z} \frac{1}{(X(z)-x)(X(z')-X(z))}\Big(\delta_z\delta_{z'}\mathcal{T}[\mathcal{S}] - \frac{\mathrm{d}X(z)\,\mathrm{d}X(z')}{(X(z)-X(z'))^2}\Big) = Q(x)$$

where Q is a rational function of x, whose only poles are those of $Y\,\mathrm{d}X$, and with degree one less than that of $Y\,\mathrm{d}X$.

Those loop equations can be written in terms of correlators by applying $\delta_{z_2}\cdots\delta_{z_n}$:

Theorem 6.2 *Let $J = \{z_2,\ldots,z_n\}$. The correlators satisfy*

(i) The linear loop equations. For all $n \geq 1$:

$$\oint_{z\in\Gamma} \frac{1}{X(z)-x}\Big(W_n(z,J) - \delta_{n,1}\,N\,(Y\mathrm{d}X)(z) - \delta_{n,2}\,\frac{\mathrm{d}X(z)\,\mathrm{d}X(z_2)}{(X(z)-X(z_2))^2}\Big) = 0$$

(ii) The quadratic loop equations. For all $n \geq 1$:

$$\oint_{z\in\Gamma} \operatorname*{Res}_{z'\to z} \frac{1}{(X(z)-x)(X(z')-X(z))}\Big\{ \sum_{I\subseteq J} W_{1+|I|}(z,I)\,W_{n-|I|}(z',J\setminus I) + \frac{1}{N^2}\,W_{n+1}(z,z',J) + \mathrm{d}X(z)\,\mathrm{d}X(z') \sum_{z_k\in J} \mathrm{d}_{z_k}\Big(\frac{W_{n-1}(J)}{(x-X(z_k))\,\mathrm{d}X(z_k)}\Big)\Big\} = Q_n(x;J)$$

defines a quantity $Q_n(x;J)$ which is a rational function of x, whose only poles are located at those of $Y\mathrm{d}X$.

The important information in loop equations, is that those particular combinations of W_n's have no monodromies in the variable x. Here in particular, they are regular at the ramification points a_i. Since every W_n has poles at ramification points to all orders in $1/N$, this is a highly non-trivial property.

Proof. The formula:

$$\mathcal{T}(\mathcal{S}) = \sum_{\mathbf{n}\in\mathbb{Z}^{\mathfrak{g}}} e^{2i\pi\mathbf{n}\cdot\nu}\, Z[\mathcal{S},\mathbf{n}] \tag{6.9}$$

makes sense, provided we collect all terms of a given order in $N\to\infty$ as in Definition 5.3. The F_g were precisely introduced such that for any $\mathbf{n}$,

$$Z[\mathcal{S},\mathbf{n}] = \mathrm{e}^{\sum_g N^{2-2g}\, F_g(\mathcal{S},\mathbf{n}+\mu)} \tag{6.10}$$

is a solution of the loop equations. By linearity, $\mathcal{T}(\mathcal{S})$ satisfy the same loop equations. In fact, the definition 5.3 of $\mathcal{T}(\mathcal{S})$ was introduced in [Eyn09] precisely to give a solution to those loop equations, which has the extra property of being independent of a choice of filling fractions. $\square$

7 Hirota equations

In the dispersionless case, we have seen that the self-replication property for ψ_{cl} is equivalent to an infinitesimal Fay identity, which is known in turn to be equivalent to Hirota equations. In a similar way, we conjecture here:

Conjecture 7.1 *ψ is self-replicating:*

$$\frac{1}{N}\,(\delta_z\psi)(z_1,z_2) = -\psi(z_1,z)\psi(z,z_2)$$

We have not been able to prove this conjecture. We prove in appendix A that it holds up to $o(1/N)$. We argue in Section 9 that it is compatible with what is known for spectral curves coming from the one matrix model (hyperelliptic curves), or the two matrix model. Besides, these matrix models do not allow to reach all plane curves $\mathcal{S}$. The difficulty in finding a proof of Conjecture 7.1 comes from the singularities at ramification points. For instance, one can always write:

$$\frac{1}{N}\,(\delta_z\psi)(z_1,z_2\,;\,\mathcal{S}) = \operatorname*{Res}_{z'\to z}\psi(z,z'\,;\,\mathcal{S})\,\psi(z',z\,;\,\mathcal{S}+[z_1]-[z_2]) \tag{7.1}$$

where $\mathcal{S}+[z_1]-[z_2]$ is the spectral curve $\mathcal{S}$ where $Y\,\mathrm{d}X$ is replaced by $Y\,\mathrm{d}X+\frac{1}{N}\,\mathrm{d}S_{z_1,z_2}$. Since the integrand is a differential form on the Riemann surface Σ underlying $\mathcal{S}$, we can move the contour to the poles at the ramification points, and the pole at $z'=z_2$:

$$\begin{aligned} & \frac{1}{N}\,(\delta_z\psi)(z_1,z_2\,;\,\mathcal{S}) \\ = & -\psi(z_1,z\,;\,\mathcal{S})\psi(z,z_2\,;\,\mathcal{S})-\sum_i \mathop{\mathrm{Res}}_{z'\to a_i}\psi(z,z'\,;\,\mathcal{S})\,\psi(z',z\,;\,\mathcal{S}+[z_1]-[z_2]) \end{aligned} \tag{7.2}$$

Then, it remains to show that the sum of residues at ramification points vanishes. So, conjecture 7.1 is equivalent to:

Conjecture 7.2

$$\sum_i \mathop{\mathrm{Res}}_{z'\to a_i}\psi(z,z'\,;\,\mathcal{S})\,\psi(z',z\,;\,\mathcal{S}+[z_1]-[z_2])=0. \tag{7.3}$$

In Appendix A, we check that this residue at each a_i is $o(1/N)$. This involves already non-trivial identities between Theta functions associated to a Riemann surface, like Fay identity or its degenerations, and involves the precise value of $\omega_3^{(0)}$. We have yet not been able to find a general way to show that this residue is 0 to all orders in $1/N$.

In terms of the $\mathcal{T}$ function, Conjecture 7.1 can be rephrased:

Conjecture 7.3 *$\mathcal{T}$ satisfy a "differential Fay identity":*

$$\mathcal{T}[\mathcal{S}]\,(\delta_z\mathcal{T})[\mathcal{S}+[z_1]-[z_2]]-(\delta_z\mathcal{T})[\mathcal{S}]\,\mathcal{T}[\mathcal{S}+[z_1]-[z_2]]=-N\,\mathcal{T}[\mathcal{S}+[z_1]-[z]]\,\mathcal{T}[\mathcal{S}+[z]-[z_2]]$$

There is also a global version of the former conjecture. First, notice from our definition in Section 5 that:

$$\mathcal{T}[(\mathcal{S}+[z_1]-[z_2])+[z_3]-[z_4]]=\mathcal{T}[(\mathcal{S}+[z_3]-[z_4])+[z_1]-[z_2]]$$

so omitting the parentheses makes sense, but:

$$\mathcal{T}[\mathcal{S}+[z_1]-[z_4]+[z_3]-[z_2]]=-\mathcal{T}[\mathcal{S}+[z_1]-[z_2]+[z_3]-[z_4]]$$

This sign comes from the fact that the definition of $\mathcal{T}$ contains a regularization procedure (for $\int_{z_i}^{z_j}\int_{z_i}^{z_j}B$), whose result depends on the way we form the pairs of simple poles to add to $\mathcal{S}$.

Conjecture 7.4 *$\mathcal{T}$ satisfies a "Fay identity":*

$$\begin{aligned} & \mathcal{T}[(\mathcal{S}+[z_1]-[z_2])+[z_3]-[z_4]]\,\mathcal{T}[\mathcal{S}] \\ = & \ \mathcal{T}[\mathcal{S}+[z_1]-[z_2]]\,\mathcal{T}[\mathcal{S}+[z_3]-[z_4]]-\mathcal{T}[\mathcal{S}+[z_3]-[z_2]]\,\mathcal{T}[\mathcal{S}+[z_1]-[z_4]] \end{aligned}$$

Provided our conjectures hold, $\mathcal{T}$ and ψ are actually the Tau function and the Baker-Akhiezer spinor kernel of a dispersive integrable system.

Proof of equivalence of Conjectures 7.3 and 7.4.

We can obtain Conjecture 7.3 from Conjecture 7.4 by letting z_1 and z_2 merge to a point z. In the other direction, we use condensed notations

$$\mathcal{T}_{ijkl} = \mathcal{T}[\mathcal{S} + [z_i] - [z_j] + [z_k] - [z_l]], \qquad \mathcal{T}_{ij} = \mathcal{T}[\mathcal{S} + [z_i] - [z_j]], \qquad \mathcal{T} = \mathcal{T}[\mathcal{S}]$$

and we apply Conjecture 7.3 to the spectral curve $\mathcal{S} + [z_3] - [z_4]$:

$$\begin{aligned} \delta_z \mathcal{T}_{1234} &= \frac{\delta_z \mathcal{T}_{34}}{\mathcal{T}_{34}}\, \mathcal{T}_{1234} - N \frac{\mathcal{T}_{1z34}\mathcal{T}_{z234}}{\mathcal{T}_{34}} \\ &= \frac{\delta_z \mathcal{T}}{\mathcal{T}}\, \mathcal{T}_{1234} - N \frac{\mathcal{T}_{3z}\mathcal{T}_{z4}\mathcal{T}_{1234}}{\mathcal{T}\mathcal{T}_{34}} - N \frac{\mathcal{T}_{1z34}\mathcal{T}_{z234}}{\mathcal{T}_{34}} \end{aligned} \tag{7.4}$$

Exchanging the roles of $z_1 \leftrightarrow z_3$ and $z_2 \leftrightarrow z_4$ also gives

$$\delta_z \mathcal{T}_{1234} = \frac{\delta_z \mathcal{T}}{\mathcal{T}}\, \mathcal{T}_{1234} - N \frac{\mathcal{T}_{1z}\mathcal{T}_{z2}\mathcal{T}_{1234}}{\mathcal{T}\mathcal{T}_{12}} - N \frac{\mathcal{T}_{3z12}\mathcal{T}_{z412}}{\mathcal{T}_{12}} \tag{7.5}$$

and comparing the two, we may get rid of the terms involving δ_z:

$$\mathcal{T}_{12}\mathcal{T}_{3z}\mathcal{T}_{z4}\mathcal{T}_{1234} + \mathcal{T}\mathcal{T}_{12}\mathcal{T}_{1z34}\mathcal{T}_{z234} \;=\; \mathcal{T}_{34}\mathcal{T}_{1z}\mathcal{T}_{z2}\mathcal{T}_{1234} + \mathcal{T}\mathcal{T}_{34}\mathcal{T}_{3z12}\mathcal{T}_{z412} \tag{7.6}$$

Let us define

$$U_{1234} = \mathcal{T}\mathcal{T}_{1234} - \mathcal{T}_{12}\mathcal{T}_{34} + \mathcal{T}_{14}\mathcal{T}_{32} \tag{7.7}$$

which is the quantity which should vanish at the end of our computation. Notice that

$$\lim_{z_1 \to z_2} \mathcal{U}_{1234} = 0 \tag{7.8}$$

Indeed:

$$\begin{aligned} \mathcal{U}_{1234} &\sim_{z_1 \to z_2} \frac{\mathcal{T}\mathcal{T}_{34}}{E_{12}} + \mathcal{T}\delta_1 \mathcal{T}_{34} - \frac{\mathcal{T}\mathcal{T}_{34}}{E_{12}} - \delta_1 \mathcal{T}\, \mathcal{T}_{34} + \mathcal{T}_{14}\mathcal{T}_{31} \\ &\sim_{z_1 \to z_2} \mathcal{T}\delta_1 \mathcal{T}_{34} - \delta_1 \mathcal{T}\, \mathcal{T}_{34} + \mathcal{T}_{14}\mathcal{T}_{31} \end{aligned} \tag{7.9}$$

where $E_{ij} = E(z_i, z_j)$, and this expression vanish by application of Conjecture 7.3 to the spectral curve $\mathcal{S}$. Notice also that the remark about $\mathcal{T}$ made above Conjecture 7.4 implies $\mathcal{U}_{ijkl} = \mathcal{U}_{klij} = -\mathcal{U}_{ilkj}$. Let us rewrite Eqn. 7.6 in terms of $\mathcal{U}_{ijkl}$ only:

$$\begin{aligned} & \mathcal{T}_{12}\mathcal{U}_{1z34}\mathcal{U}_{z234} - \mathcal{T}_{34}\mathcal{U}_{3z12}\mathcal{U}_{z412} + (\mathcal{T}_{12}\mathcal{T}_{3z}\mathcal{T}_{z4} - \mathcal{T}_{1z}\mathcal{T}_{z2}\mathcal{T}_{34})\mathcal{U}_{1234} \\ & +(\mathcal{T}_{1z}\mathcal{T}_{34} - \mathcal{T}_{14}\mathcal{T}_{3z})\mathcal{T}_{12}\mathcal{U}_{z234} + (\mathcal{T}_{z2}\mathcal{T}_{34} - \mathcal{T}_{32}\mathcal{T}_{z4})\mathcal{T}_{12}\mathcal{U}_{1z34} \\ & +(\mathcal{T}_{32}\mathcal{T}_{1z} - \mathcal{T}_{12}\mathcal{T}_{3z})\mathcal{T}_{34}\mathcal{U}_{z412} + (\mathcal{T}_{z2}\mathcal{T}_{14} - \mathcal{T}_{12}\mathcal{T}_{z4})\mathcal{T}_{34}\mathcal{U}_{3z12} \\ = \;& 0 \end{aligned}$$

The left hand side may have simple poles when $z_i \to z_j$ due to $\mathcal{T}_{ij}$, but not higher degree poles since we know that $\mathcal{U}_{ijkl}$ is actually regular when $z_i \to z_j$ or z_l. For Eqn. 7.10 to hold, in particular, the coefficient of the simple pole when $z_2 \to z_3$ must vanish:

$$-\mathcal{T}_{13}\mathcal{T}_{z4}\mathcal{U}_{1z34} + \mathcal{T}_{1z}\mathcal{T}_{34}\mathcal{U}_{z413} = 0 \tag{7.10}$$

which we can also write after reindexing the points and using the symmetries of $\mathcal{U}$:

$$\mathcal{T}_{31}\mathcal{T}_{z4}\mathcal{U}_{1z34} = \mathcal{T}_{3z}\mathcal{T}_{14}\mathcal{U}_{z134} \tag{7.11}$$

Similarly, the coefficient of the simple pole when $z_1 \to z_2$ must vanish:

$$\mathcal{U}_{1z34}\mathcal{U}_{z134} + (\mathcal{T}_{z1}\mathcal{T}_{34} - \mathcal{T}_{z4}\mathcal{T}_{31})\mathcal{U}_{1z34} + (\mathcal{T}_{1z}\mathcal{T}_{34} - \mathcal{T}_{14}\mathcal{T}_{3z})\mathcal{U}_{z134} \quad = \quad 0 \tag{7.12}$$

Now, we may combine the latter with Eqn. 7.11, set $z = z_2$ for convenience, and isolate $\mathcal{U}_{1234}$:

$$\begin{aligned}&\frac{\mathcal{T}_{31}\mathcal{T}_{24}}{\mathcal{T}_{32}\mathcal{T}_{14}}\mathcal{U}_{1234}^2\\ &\Big((\mathcal{T}_{21}\mathcal{T}_{34} - \mathcal{T}_{24}\mathcal{T}_{31}) + \frac{\mathcal{T}_{31}\mathcal{T}_{24}}{\mathcal{T}_{32}\mathcal{T}_{14}}(\mathcal{T}_{12}\mathcal{T}_{34} - \mathcal{T}_{14}\mathcal{T}_{32})\Big)\mathcal{U}_{1234} = 0\end{aligned} \tag{7.13}$$

If $\mathcal{U}_{1234}$ were not identically zero, we would have (by continuity of all the coefficients of the series) for any points z_1, z_2, z_3, z_4 on the curve:

$$\mathcal{U}_{1234} = \frac{\mathcal{T}_{32}\mathcal{T}_{14}}{\mathcal{T}_{31}\mathcal{T}_{24}}(\mathcal{T}_{24}\mathcal{T}_{31} - \mathcal{T}_{21}\mathcal{T}_{34}) + (\mathcal{T}_{14}\mathcal{T}_{32} - \mathcal{T}_{12}\mathcal{T}_{34}) \tag{7.14}$$

But matching the coefficient of the simple pole when $z_1 \to z_4$ in this equation yields:

$$0 = \frac{\mathcal{T}_{32}}{\mathcal{T}_{31}\mathcal{T}_{21}}(\mathcal{T}_{21}\mathcal{T}_{31} - \mathcal{T}_{21}\mathcal{T}_{31}) + \mathcal{T}_{32} = \mathcal{T}_{32} \tag{7.15}$$

which is not true. Therefore, $\mathcal{U}_{1234} \equiv 0$. $\square$

8 Consequences

8.1 Exponential formula

We start with a remark which does not rely of the conjectures of Section 7, but which is natural to present now. Recall that the kernel is defined by Sato's formula:

$$\psi(z_1, z_2\,;\,\mathcal{S}) = \frac{\mathcal{T}(\mathcal{S}, Y\mathrm{d}X \to Y\mathrm{d}X + \frac{1}{N}\,\mathrm{d}S_{z_1,z_2})}{\mathcal{T}(\mathcal{S})}\,\sqrt{\mathrm{d}X(z_1)\,\mathrm{d}X(z_2)} \tag{8.1}$$

Adding a double pole to $Y\mathrm{d}X$ can be realized by adding two simple poles and take the limit where the two simple poles collapse. In other words, we can write $\mathrm{d}S_{z_1,z_2} = \int_{z_2}^{z_1} B = (X(z_1)-X(z_2))\, B(.,z_1)/\mathrm{d}X(z_1)+O((X(z_1)-X(z_2)))^2$, and then express ψ in terms of second kind deformations of $\mathcal{T}$, i.e. in terms of the correlators W_n. However, when we substitute second kind deformations instead of third kind deformations, we must pay attention to regularization for the term $\int_2^1\int_2^1 B$, like in Section 4.1 and 5.3. Besides, for any n, W_n has a finite number of terms which are not $O(1/N)$. For this reason we define the following quantities, which are $O(1/N)$:

$$\widehat{W}_1(z) = W_1(z) - N\,Y\mathrm{d}X(z) - (\ln\Theta)'\cdot 2i\pi\,\mathrm{d}\mathbf{u}(z) \tag{8.2}$$

$$\widehat{W}_2(z_1,z_2) = W_2(z_1,z_2) - B(z_1,z_2) - (\ln\Theta)''\cdot 2i\pi\,\mathrm{d}\mathbf{u}(z_1)\otimes 2i\pi\,\mathrm{d}\mathbf{u}(z_2) \tag{8.3}$$

and for $n\geq 3$:

$$\widehat{W}_n(z_1,\dots,z_n) = W_n(z_1,\dots,z_n) - (\ln\Theta)^{(n)}\cdot\bigotimes_{j=1}^{n} 2i\pi\,\mathrm{d}\mathbf{u}(z_j) \tag{8.4}$$

Then, we have the exponential formula:

Proposition 8.1

$$\psi(z_1,z_2) = \frac{e^{N\int_{z_2}^{z_1} Y\mathrm{d}X}}{E(z_1,z_2)}\,\frac{\Theta_{12}}{\Theta}\,\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\int_{z_2}^{z_1}\cdots\int_{z_2}^{z_1}\widehat{W}_n\Big)$$

where

$$\Theta_{12} = \Theta(\mathbf{w}_0 + 2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau) \qquad , \quad \Theta = \Theta(\mathbf{w}_0|\,\tau). \tag{8.5}$$

This formula is an equality if we collect on both sides all the terms of the same order.

Proof. The Taylor formula allows to express $\ln\mathcal{T}[\mathcal{S}+[z_1]-[z_2]]-\ln\mathcal{T}[\mathcal{S}]$ to all orders in $1/N$. For the first and second order, we use Eqn. 4.2 and Eqn. 4.3.

$$\begin{aligned}
& \ln\mathcal{T}[\mathcal{S}+[z_1]-[z_2]] - \ln\mathcal{T}[\mathcal{S}] \\
= \;& \int_{z_2}^{z_1} N^{-1}\delta_\zeta(\ln\mathcal{T}) + \frac{1}{2}\text{ ''}\int_{z_2}^{z_1}\int_{z_2}^{z_1} N^{-2}\delta_\zeta\delta_{\zeta'}(\ln\mathcal{T})[\mathcal{S}]\text{ ''} \\
& +\sum_{n\geq 3}\frac{N^{-n}}{n!}\int_{z_2}^{z_1}\underbrace{\delta_{\zeta_1}\cdots\int_{z_2}^{z_1}\delta_{\zeta_n}}_{n\text{ times}}(\ln\mathcal{T})[\mathcal{S}] \\
= \;& \int_{z_2}^{z_1} W_1 + \frac{1}{2}\text{ ''}\int_{z_2}^{z_1}\int_{z_2}^{z_1} W_2\text{ ''} \\
& +\sum_{n\geq 3}\frac{1}{n!}\underbrace{\int_{z_2}^{z_1}\cdots\int_{z_2}^{z_1}}_{n\text{ times}} W_n
\end{aligned}$$

$$
\begin{aligned}
&= N\int_{z_2}^{z_1} Y\mathrm{d}X - \frac{1}{2}\ln\Big((E(z_1,z_2))^2\mathrm{d}X(z_1)\mathrm{d}X(z_2)\Big) \\
&\quad + \sum_{n\geq 1}\frac{1}{n!}(\ln\Theta)^{(n)}\cdot\left(2i\pi\left(\mathbf{u}(z_1)-\mathbf{u}(z_2)\right)\right)^{\otimes n} \\
&\quad + \sum_{n\geq 1}\frac{1}{n!}\underbrace{\int_{z_2}^{z_1}\cdots\int_{z_2}^{z_1}}_{n\text{ times}}\widehat{W}_n
\end{aligned}
\tag{8.6}
$$

In the last step, we have used the expression for the leading order of W_n found in Section 6.3. Then, the second line can be resummated into $\ln\Theta_{12}$, and the whole result exponentiated leads to the formula we announced. $\square$

Notice that, when expanding the exponential, to any given order $O(N^{-k})$, all the W_n give a contribution involving derivatives of theta functions contracted with tensor products $(\mathbf{u}(z_1)-\mathbf{u}(z_2))^{\otimes n}$. These contributions have to be resummated into a single theta function (or derivatives thereof) with argument shifted by $\mathbf{u}(z_1)-\mathbf{u}(z_2)$, then producing an expression at the order sought involving only a finite number of terms.

8.2 Determinantal formulas

Conversely, the correlators $W_n(z_1,\ldots,z_n)$ can be recovered from the spinor kernel $\psi(z_1,z_2)$, they are the determinantal correlation functions built with $\psi(z_i,z_j)$. Let us consider first the case of W_1, which does not rely on the conjectures of Section 7.

Lemma 8.1

$$
W_1(z) = N\,Y\mathrm{d}X(z) + \lim_{z'\to z}\left(\psi(z',z)e^{-N\int_{z_2}^{z_1}Y\mathrm{d}X} - \frac{\sqrt{\mathrm{d}X(z')\,\mathrm{d}X(z)}}{X(z')-X(z)}\right)
$$

Proof. First, we notice that adding a double pole to $Y\mathrm{d}X$ can be realized by adding two simple poles and take the limit where the two simple poles collapse. More precisely, when $z'\to z$, we have

$$
\mathrm{d}S_{z',z}(z_0) \underset{z'\to z}{\sim} (X(z')-X(z))\,\frac{B(z,z_0)}{\mathrm{d}X(z)}
\tag{8.7}
$$

We can thus use the definition of δ_z with $\lambda = X(z')-X(z)$. For any regular functional $f(Y\mathrm{d}X)$ of the spectral curve, we thus have:

$$
f\Big(Y\mathrm{d}X+\frac{1}{N}\mathrm{d}S_{z',z}\Big) = f(Y\mathrm{d}X)+\frac{X(z')-X(z)}{N\,\mathrm{d}X(z)}\,\delta_z f(Y\mathrm{d}X)+O((X(z')-X(z))^2)
\tag{8.8}
$$

In some sense we trade a variation of $Y\mathrm{d}X$ by a second kind differential with a variation with a third kind differential.

In particular, F_g with $g \geq 1$, or every Θ-term in Definition 5.3 are regular functionals of the spectral curve. We just have to pay attention to the F_0 term, because the derivative of F_0 with respect to third kind differentials involves a regularization procedure, whereas the derivative with respect to second kind differentials does not. We have, by Taylor expansion, and computing all derivatives from special geometry (Theorem 5.2):

$$\begin{aligned}
F_0(Y\mathrm{d}X + \lambda\,\mathrm{d}S_{z',z}) &= F_0(Y\mathrm{d}X) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!}\,\left.\frac{\partial^n}{\partial\lambda^n} F_0(Y\mathrm{d}X + \lambda\,\mathrm{d}S_{z',z})\right|_{\lambda=0} \\
&= F_0(Y\mathrm{d}X) + \lambda \int_z^{z'} Y\mathrm{d}X \\
&\quad - \frac{\lambda^2}{2} \ln\left(E(z,z')^2 \mathrm{d}X(z)\mathrm{d}X(z')\right) \\
&\quad + \sum_{n=3}^{\infty} \frac{\lambda^n}{n!} \int_z^{z'} \cdots \int_z^{z'} W_n^{(0)} \\
&= F_0(Y\mathrm{d}X) + \lambda \int_z^{z'} Y\mathrm{d}X \\
&\quad - \frac{\lambda^2}{2} \ln\big(E(z,z')^2 \mathrm{d}X(z)\mathrm{d}X(z')\big) \\
&\quad + O((X(z') - X(z))^2)
\end{aligned} \tag{8.9}$$

Taking $\lambda = 1/N$ gives:

$$\begin{aligned}
& F_0(Y\mathrm{d}X + \frac{1}{N}\,\mathrm{d}S_{z',z}) + \frac{1}{N^2} \ln\left(E(z',z)\,\sqrt{\mathrm{d}X(z)\mathrm{d}X(z')}\right) \\
= \;& F_0(Y\mathrm{d}X) + \frac{X(z') - X(z)}{N\,\mathrm{d}X(z)}\,\delta_z F_0(Y\mathrm{d}X) \\
& + O((X(z') - X(z))^2)
\end{aligned} \tag{8.10}$$

Finally we have:

$$\psi(z',z)\,\mathrm{e}^{-N\int_z^{z'} Y\mathrm{d}X}\,E(z',z) = 1 + \frac{X(z') - X(z)}{N\,\mathrm{d}X(z)}\,\delta_z\left(\ln\mathcal{T} - N^2 F_0\right) + O((X(z') - X(z))^2) \tag{8.11}$$

and thus

$$\psi(z',z)\,\mathrm{e}^{-N\int_z^{z'} Y\mathrm{d}X} - \frac{1}{E(z',z)} = \frac{X(z') - X(z)}{N\,\mathrm{d}X(z)\,E(z',z)}\,\delta_z\left(\ln\mathcal{T} - N^2 F_0\right) + O(X(z') - X(z)) \tag{8.12}$$

Taking the limit $z' \to z$, and noticing that $1/E(z',z) = \sqrt{\mathrm{d}X(z)\,\mathrm{d}X(z')}/(X(z') - X(z)) + O(X(z') - X(z))$, gives the lemma:

$$\frac{1}{N}\,\delta_z\left(\ln\mathcal{T} - N^2 F_0\right) \;=\; W_1(z) - N\,Y(z)\mathrm{d}X(z)$$

$$= \lim_{z' \to z} \left(\psi(z', z)\, \mathrm{e}^{-N \int_z^{z'} Y \mathrm{d}X} - \frac{\sqrt{\mathrm{d}X(z')\mathrm{d}X(z)}}{X(z') - X(z)} \right) \quad (8.13)$$

□

Theorem 8.1 *If Conjecture 7.1 holds, then*

$$\forall n \geq 2, \quad W_n(z_1, \ldots, z_n) = (-1)^{n+1} \sum_{\sigma \,\text{cyclic perm.}} \prod_{i=1}^{n} \psi(z_i, z_{\sigma(i)})$$

Equivalently:

$$\overline{W}_n(z_1, \ldots, z_n) = \text{"}\det\text{"}\, \psi(z_i, z_j)$$

where " det " *means that when we decompose the determinant as a sum of permutations, each factor* $\psi(z_i, z_i)$ *should be replaced by* $W_1(z_i)$.

Proof. The formula for W_1 is proved in Lemma 8.1. Then, we get the formula for W_n by recursively applying δ_{z_i} and using the self-replication of ψ. □

It is thus clear that this determinantal structure relies on the (conjectured) existence of Hirota equations for $\mathcal{T}$, i.e. on integrability.

8.3 Baker-Akhiezer functions

We recall the notations of Section 3. With the kernel $\psi(z_1, z_2)$ of our dispersive construction, we introduce a $d \times d$ matrix $\Psi(x_1, x_2\,; \mathcal{S}) = \psi(z^i(x_1), z^j(x_2)\,; \mathcal{S})_{i,j=1,\ldots,d}$ for $x_1, x_2 \in \mathbb{C}$. This definition has to be regularized when x_1 or x_2 is equal to $X(p)$ where $p \in \overline{\mathcal{P}}$. In particular, it was explained in Section 3.4 how to take x_1 or $x_2 \to \infty$:

$$\Psi(x\,; \mathcal{S}) \quad \text{"} = \text{"} \quad [\Psi(z^j(x), \infty_I\,; \mathcal{S})]_{I,1 \leq j \leq d} \quad (8.14)$$

$$\Phi(x\,; \mathcal{S}) \quad \text{"} = \text{"} \quad [\Psi(\infty_I, z^j(x))\,; \mathcal{S}]_{I,1 \leq j \leq d} \quad (8.15)$$

8.4 Duality equation

Theorem 8.2 *If Conjecture 7.1 holds, we have for any spectral curve* $\mathcal{S}$*:*

$$\Psi(x_1, x_2\,; \mathcal{S})\Psi(x_2, x_3\,; \mathcal{S}) = \frac{(x_1 - x_3)\,\mathrm{d}x_2}{(x_2 - x_1)(x_3 - x_1)} \Psi(x_1, x_3\,; \mathcal{S})$$

Proof. For $\mathcal{S} = (\mathcal{C}, x, y)$, we introduce an auxiliary spectral curve:

$$\hat{\mathcal{S}}_{ij} = \left(\mathcal{C}, X, Y + \frac{1}{N} \frac{\mathrm{d}S_{z^i(x_1), z^j(x_3)}}{\mathrm{d}x} \right) \quad (8.16)$$

To compute the matrix element, we use the self-replication of ψ:

$$\begin{aligned}
[\Psi(x_1,x_2\,;\,\mathcal{S})\Psi(x_2,x_3\,;\,\mathcal{S})]_{ij} &= \sum_m \psi(z^i(x_1),z^m(x_2)\,;\,\mathcal{S})\psi(z^m(x_2),z^j(x_3)\,;\,\mathcal{S}) \\
&= -\sum_m \delta_{z^m(x_2)} \ln\left(\frac{\mathcal{T}[\hat{\mathcal{S}}_{ij}]}{\mathcal{T}[\mathcal{S}]}\right)
\end{aligned} \tag{8.17}$$

Now, we use the fact that δ_z is a derivation, and by definition, $\delta_z \ln \mathcal{T}[\hat{\mathcal{S}}_{ij}] = W_1[z\,;\,\hat{\mathcal{S}}_{ij}]$.

$$\begin{aligned}
[\Psi(x_1,x_2\,;\,\mathcal{S})\Psi(x_2,x_3\,;\,\mathcal{S})]_{ij} &= -\sum_m W_1[z^m(x_2)\,;\,\hat{\mathcal{S}}_{ij}]\,\psi(z^i(x_1),z^j(x_3)\,;\,\mathcal{S}) \\
&\quad +\sum_m W_1[z^m(x_2)\,;\,\mathcal{S}]\,\psi(z^i(x_1),z^j(x_3)\,;\,\mathcal{S})
\end{aligned} \tag{8.18}$$

The linear loop equation (Theorem 6.1) tells us the sum over sheets of $W_1[z^m(x_2)\,;\,\hat{\mathcal{S}}_{ij}]$:

$$\begin{aligned}
&[\Psi(x_1,x_2\,;\,\mathcal{S})\Psi(x_2,x_3\,;\,\mathcal{S})]_{ij} \\
= \;& [\Psi(x_1,x_3\,;\,\mathcal{S})]_{ij}\Big(-\sum_m N\,Y\mathrm{d}X(z^m(x_2)) - \mathrm{d}S_{z^i(x_1),z^j(x_3)}(z^m(x_2)) \\
&\qquad\qquad + N\,Y\mathrm{d}X(z^m(x_2))\Big) \\
= \;& -[\Psi(x_1,x_3\,;\,\mathcal{S})]_{ij}\sum_m \mathrm{d}S_{z^i(x_1),z^j(x_3)}(z^m(x_2)) \\
= \;& \frac{(x_1-x_3)\,\mathrm{d}x_2}{(x_1-x_2)(x_2-x_3)}\,[\Psi(x_1,x_3\,;\,\mathcal{S})]_{ij}
\end{aligned} \tag{8.19}$$

□ The spinor kernel $\psi_{\mathrm{cl}}(z_1,z_2)$ of the dispersionless integrable system of Section 3 was regular at ramification points. So, we could find a formula for $\psi_{cl}(z_1,z)\psi_{\mathrm{cl}}(z,z_2)$ even before the sum over the sheets where z is located (refined duality equation, Theorem 3.3). Here, the spinor kernel $\psi(z_1,z_2)$ does have, order by order in $1/N$, poles at ramification points. So, we do not have a simple expression for $\psi(z_1,z)\psi(z,z_2)$. However, Theorem 8.2 shows all contributions from the ramification points cancel in the sum over sheets.

8.5 Christoffel-Darboux relations

The matrix $\Psi(x_1,x_2\,;\,\mathcal{S})$ is invertible, since it is defined as a series for which the leading term coincides with $\Psi_{\mathrm{cl}}(x_1,x_2\,;\,\mathcal{S})$ which is invertible (see Lemma 3.2). We also have a duality relation to express the inverse:

Corollary 8.1 *If Conjecture 7.1 holds,*

$$\Psi^{-1}(x_1,x_2\,;\,\mathcal{S}) = -\frac{(x_1-x_2)^2}{\mathrm{d}x_1\mathrm{d}x_2}\,\Psi(x_2,x_1\,;\,\mathcal{S})$$

Theorem 8.3 *If Conjecture 7.1 holds, the matrices $\Psi(x\,;\mathcal{S})$ and $\Phi(x\,;\mathcal{S})$ are invertible, and we have the Christoffel–Darboux relation:*

$$\psi(z_1,z_2\,;\mathcal{S})=\frac{\sum_{I,J}\psi_I(z_1\,;\mathcal{S})A_{I,J}[\mathcal{S}]\phi_J(z_2\,;\mathcal{S})}{X(z_1)-X(z_2)}$$

where the matrix A is invertible, independent of x and given by:

$$A^{-1}[\mathcal{S}]=\frac{1}{\mathrm{d}x}\Phi(x\,;\mathcal{S})\Psi^t(x\,;\mathcal{S})$$

The proofs can be copied from the classical case (Section 3.5), because they are based only on the duality equation.

8.6 Differential systems

$\Psi(x_1,x_2\,;\mathcal{S})$ is the solution of a system of differential equations with respect to the positions of the poles $X(p)$ ($x=\infty$ is a fixed pole) and the times $t_{p,j}$.

Theorem 8.4 *If Conjecture 7.1 holds, then for any deformation parameter $\lambda=t_{p,j}$ or $\lambda=X(p)\neq\infty$, there exists a $d\times d$ matrix $M_\lambda(x_1,x_2\,;\mathcal{S})$, such that:*

(i) $(\partial_\lambda-M_\lambda)\,\Psi(x_1,x_2\,;\mathcal{S})=0$.

(ii) *M_λ is a rational function of x_2.*

(iii) *M_λ has no pole at branchpoints.*

These deformations are compatible (since Ψ is invertible) and isomonodromic.

Implicitly, for $j=0$, only deformations $\partial_\lambda=\partial_{t_{p,0}}-\partial_{t_{p',0}}$ are considered.

Proof. By definition:

$$\begin{aligned} M_\lambda &= \partial_\lambda\Psi(x_1,x_2)\,\Psi^{-1}(x_1,x_2) \\ &= -\frac{(x_1-x_2)^2}{\mathrm{d}x_1\mathrm{d}x_2}\partial_\lambda\Psi(x_1,x_2)\,\Psi(x_2,x_1) \end{aligned} \tag{8.20}$$

When λ is a time $t_{p,k}$ ($k\geq 1$), we have by self-replication:

$$\partial_{t_{p,k}}\psi(z_1,z_2)=-\int_{\omega^*_{p,k}}\psi(z_1,\cdot)\psi(\cdot,z_2) \tag{8.21}$$

We compute:

$$\begin{aligned} [M_{t_{p,k}}]_{ij} &= \frac{(x_1-x_2)^2}{\mathrm{d}x_1\mathrm{d}x_2}\sum_m\int_{\omega^*_{p,k}}\psi(z_1^i,z)\psi(z,z_2^m)\psi(z_2^m,z_1^j) \\ &= \frac{1}{\mathrm{d}x_1}\int_{\omega^*_{p,k}}\frac{(X(\cdot)-x_1)(x_2-x_1)}{(X(\cdot)-x_2)}\psi(z_1^i,\cdot)\psi(\cdot,z_1^j) \end{aligned} \tag{8.22}$$

It is clear that $M_{t_{p,k}}(x_1, x_2)$ is a rational function of x_2 without poles at branchpoints. If we assume $X(p) \neq \infty$, it has a pole only at $x_2 = X(p)$, and this pole is of order $k+1$. If we assume $X(p) = \infty$, it has a pole only at $x_2 = \infty$, and this pole is of order $1 + \lfloor k/d_p \rfloor$ (we recall that d_p is the multiplicity of order of the pole p of X). It is straightforward to adapt this proof for $k = 0$, namely $\partial_\lambda = \partial_{t_{p,0}} - \partial_{t_{p',0}}$.

Now, we turn to $\lambda = X(p) \neq \infty$. We have assumed that $\mathrm{d}X(p) \neq 0$, so the preimages $p^1, \ldots, p^d$ of $X(p)$ are distinct, and $(X(z) - X(p))$ is a local coordinate near each p^m. The Laurent expansion of $Y \mathrm{d}X(z)$ when $z \to p^m$ is:

$$Y \mathrm{d}X(z) = \sum_{k \geq 0} t_{p,k} \frac{\mathrm{d}X(z)}{(X(z) - X(p))^{k+1}} + O(1) \tag{8.23}$$

where only a finite number of $t_{p,k'} = 0$ are non zero. If we perform $X(p) \to X(p) + \lambda$ while the times are fixed, we change $Y \mathrm{d}X$ to $(Y \mathrm{d}X)_\lambda$, with Laurent expansion at p^m:

$$(Y \mathrm{d}X)_\lambda = Y \mathrm{d}X + \lambda \left(\sum_{k \geq 1} \frac{k t_{p,k-1}}{(X(z) - X(p))^{k+1}} \right) + O(1) \tag{8.24}$$

So, we can identify:

$$\frac{\partial}{\partial X(p)} \longrightarrow \sum_{k \geq 1} k t_{p,k-1} \frac{\partial}{\partial t_{p,k}} \tag{8.25}$$

which is a finite sum. Hence:

$$[M_{X(p)}]_{ij} = -\frac{1}{\mathrm{d}x_1} \sum_m \operatorname*{Res}_{z \to z^m(X(p))} \frac{\mathrm{d}Y(z)}{\mathrm{d}X(z)} \frac{(X(z) - x_1)(x_2 - x_1)}{(X(z) - x_2)} \psi(z_1^i, z) \psi(z, z_1^j) \tag{8.26}$$

This is a rational fraction of x_2, with a pole at $x_2 = X(p)$ of order $1 + \lfloor (\max_m \mathrm{ord}_{z^m(p)} Y \mathrm{d}X)/d_p \rfloor$. □

One could also define a matrix L with the reconstruction formula presented in Section 3.6. But now, the function Y depends on the spectral curve, the evolution of L under the flows generated by the times is not isospectral. But the argument of Eqn. 3.8 is still valid, and the times in this construction are isomonodromic parameters. Such a reconstruction has also been performed at leading order by Bertola and Gekhtman [BG07], and our expressions up to $o(1)$ match indeed their results.

9 Parallel with matrix models

Some unitarily invariant models of random matrices are special cases of this construction, that actually suggested to develop this formalism. We recall here the dictionary between the definitions in integrability (and in this article), and observables in matrix models. Let M be a random square matrix of size $N \times N$, diagonalizable by a unitary

conjugation, with eigenvalues restricted to some contour Γ in the complex plane. The probability measure $\mathrm{d}\mu(M)$ is a data of the matrix model. Its normalization defines the partition function $Z_N = \int \mathrm{d}\mu(M)$. For a function f, we note $\left\langle f(M) \right\rangle$ the expectation value of the random variable $f(M)$:

$$\left\langle f(M) \right\rangle = \frac{\int \mathrm{d}\mu(M)\, f(M)}{\int \mathrm{d}\mu(M)} \tag{9.1}$$

9.1 Examples of integrable matrix models

Important examples of integrable matrix models, described in details in [Meh04], include:

The one hermitian matrix model

$M \in H_N$ is a $N \times N$ hermitian matrix, with eigenvalues on the real axis

$$\mathrm{d}\mu(M) = \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)}\,\mathrm{d}M \qquad , \qquad \mathrm{d}M = \prod_{i=1}^{N} \mathrm{d}M_{i,i} \prod_{1\leq i<j\leq N} \mathrm{dRe}\, M_{i,j}\ \mathrm{dIm}\, M_{i,j} \tag{9.2}$$

and $V(M)$ is a "semiclassical potential" (see [Ber07]), i.e. $V'(x) \in \mathbb{C}(x)$, i.e. $V'(x)$ is a rational function of its variable x, chosen such that $\int_{\mathbb{R}} \mathrm{e}^{-V(x)}\mathrm{d}x$ is absolutely convergent.

The one normal matrix model, with eigenvalues on a contour Γ

We define

$$H_N(\Gamma) = \{U\,\mathrm{diag}(x_1,\dots,x_N)\,U^\dagger \qquad U \in U(N) \text{ and } (x_1,\dots,x_N) \in \Gamma^N\} \tag{9.3}$$

equipped with a "measure" (not necessarily real or normalized)[7]:

$$\mathrm{d}M = \mathrm{d}U\ \prod_{i<j}(x_i - x_j)^2\ \prod_{i=1}^{N} \mathrm{d}x_i \tag{9.4}$$

where $\mathrm{d}U$ is the Haar measure on the unitary group $U(N)$ and $\mathrm{d}x_i$ is the curvilinear[8] measure along Γ. This measure is always invariant under unitary transformations. We then define the measure $\mathrm{d}\mu(M)$ on $H_N(\Gamma)$:

$$\mathrm{d}\mu(M) = \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)}\,\mathrm{d}M, \tag{9.5}$$

[7] For a given $M \in H_N(\Gamma)$, there exist many choices of U and x_i's (in fact U can be multiplied by any element of $U(1)^N$ and the x_i's can be permuted, but the reader can check that the measure $\mathrm{d}M$ is well defined on $H_N(\Gamma)$.

[8] If $\Gamma \subset \mathbb{C}$ is a C^1 path in the complex plane parametrized by a C^1 function $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{C}$, i.e. $\Gamma = \{\gamma(s)\,,\, s \in \mathbb{R}\}$, at $x = \gamma(s)$ we define the curvilinear measure $\mathrm{d}x = \gamma'(s)\,\mathrm{d}s$ where $\mathrm{d}s$ is the Lebesgue measure on $\mathbb{R}$.

where $V'(x) \in \mathbb{C}(x)$, i.e. $V'(x)$ is a rational function of its variable x, chosen such that $\int_\Gamma \mathrm{e}^{-V(x)}\mathrm{d}x$ is absolutely convergent. This means that for a given potential $V(x)$, Γ must go to ∞ or to the poles of V', only in sectors where $\mathrm{Re}\, V \to +\infty$, i.e. in some "Stokes sectors". When $\Gamma = \mathbb{R}$, this definition is the same as the Hermitian matrix model $H_N(\mathbb{R}) = H_N$, with the usual Lebesgue measure on H_N. When Γ is the unit circle $\mathbb{S}_1$, this definition is the same as the unitary matrix model $H_N(\mathbb{S}_1) = U(N)$, with the usual Haar measure on $U(N)$.

The two normal matrices model

Given two paths Γ_1, Γ_2, we define a measure on $H_N(\Gamma_1) \times H_N(\Gamma_2)$

$$\mathrm{d}\mu_2(M_1, M_2) = \mathrm{e}^{-N\,\mathrm{Tr}\,[V_1(M_1)+V_2(M_2)-M_1M_2]} \tag{9.6}$$

where V_1 and V_2 are semiclassical potentials (V_1' and V_2' are rational functions), chosen such that integrals on $\Gamma_1 \times \Gamma_2$ are absolutely convergent. Upon integration on M_2, this measure induces a measure $\mathrm{d}\mu(M)$ on M_1 that we rename M:

$$\mathrm{d}\mu(M) = \int_{M_2 \in H_N(\Gamma_2)} \mathrm{d}\mu_2(M, M_2) \tag{9.7}$$

The chain of matrices

This is the natural generalization of the case of two matrices. Consider k paths $\Gamma_1, \dots, \Gamma_k$, and k semiclassical potentials $V_1, \dots, V_k$, and define the measure on $H_N(\Gamma_1) \times \dots \times H_N(\Gamma_k)$ as

$$\mathrm{d}\mu_k(M_1, \dots, M_k) = \mathrm{e}^{-N\,\mathrm{Tr}\,[\sum_{i=1}^k V_i(M_i) - \sum_{i=1}^{k-1} M_iM_{i+1}]} \tag{9.8}$$

For any $i \in \{1, \dots, k\}$, we integrate on M_j's with $j \neq i$, and renaming $M = M_i$, this measure induces a measure $\mathrm{d}\mu(M)$:

$$\mathrm{d}\mu(M_i) = \int_{M_j \in H_N(\Gamma_j), j\neq i} \mathrm{d}\mu_k(M_1, \dots, M_k). \tag{9.9}$$

9.2 Correspondences

9.2.1 Partition function

All the listed matrix models above have the property to be integrable, in a sense explained below. There is a huge literature on the subject, let us mention among others the early works in physics [MMKZ92, Kos96], and in mathematics [HTW93, AM95, vM97]. For the one and two matrices models, it is established that the partition function $Z_N = \int \mathrm{d}\mu(M)$ is an isomonodromic Tau function [BEH03c, BEH06, BM09a]

(such a result is not known at present for the chain of matrices). The so-called double scaling limit of matrix models has been also intensively from the point of view of isomonodromic deformations [IKF90, DS90, Moo90, FIKN06].

9.2.2 Correlators, spectral curves, loop equations

For all the matrix models above, the correlators are:

$$\mathcal{W}_n(x_1,\dots,x_n) = N^{-n}\left\langle \prod_{i=1}^n \operatorname{Tr}\frac{1}{x_i-M}\right\rangle_C \tag{9.10}$$

$$\overline{\mathcal{W}}_n(x_1,\dots,x_n) = N^{-n}\left\langle \prod_{i=1}^n \operatorname{Tr}\frac{1}{x_i-M}\right\rangle \tag{9.11}$$

where C means 'cumulant'. $\mathcal{W}_n$ is often called n-point function, or *connected* n-point function, and $\overline{\mathcal{W}}_n$ is called disconnected n-point function.

The one point function $\mathcal{W}_1(x)$ plays an important role, it is often called the *resolvent*. In all the matrix models above, under reasonable assumptions, it has a large N limit denoted:

$$\frac{1}{N}\,\mathcal{W}_1(x) \underset{N\to\infty}{\sim} \mathcal{W}_1^{(0)}(x) = \mathcal{Y}(x) \tag{9.12}$$

which is furthermore an algebraic function of x, i.e. there exists a polynomial $\mathcal{E}(x,y)$ such that

$$\mathcal{E}(x,\mathcal{Y}(x)) = 0. \tag{9.13}$$

This algebraic equation defines an algebraic curve $\mathcal{C}$, i.e. a compact Riemann surface $\mathcal{C}$ and two meromorphic functions $X, Y : \mathcal{C} \to \mathbb{C}P^1$, such that

$$\forall\, z\in\mathcal{C}\,, \qquad \mathcal{Y}(X(z)) = Y(z). \tag{9.14}$$

The "semiclassical spectral curve" $\mathcal{S}$ is then

$$\mathcal{S} = (\mathcal{C}, X, Y). \tag{9.15}$$

The correlators corresponding to Definition 6.2, are

$$W_n(z_1,\dots,z_n) = N^n\,\mathcal{W}_n(X(z_1),\dots,X(z_n))\bigotimes_{i=1}^n \mathrm{d}X(z_i) + \delta_{n,2}\,\frac{\mathrm{d}X(z_1)\otimes\mathrm{d}X(z_2)}{(X(z_1)-X(z_2))^2} \tag{9.16}$$

It is a classical result of random matrix theory (for instance it can be proved by integration by parts in the matrix integral) that, for all matrix models listed above, the correlators W_n satisfy the loop equations of Theorem 6.1.

9.2.3 Large N asymptotic expansion

It is conjectured that the large N asymptotic expansion of the partition function Z_N matches with that of the $\mathcal{T}$ function for the semiclassical spectral curve $\mathcal{S}$ introduced in Definition 5.3:

$$Z_N = \mathcal{T}_{[\mu,\nu]}(\mathcal{S}) \tag{9.17}$$

i.e. has an asymptotic expansion

$$\ln Z_N = N^2 F_0 + F_1 + \ln \Theta + \frac{1}{N}\left(F_1' \frac{\Theta'}{\Theta} + F_0''' \frac{\Theta'''}{\Theta}\right) + o(1/N) \tag{9.18}$$

The characteristics $[\mu, \nu]$ of the Theta function Θ is determined by the choice of the integration contour Γ. This conjecture was derived heuristically in [Eyn09].

When the semiclassical spectral curve is of genus 0, there is no Theta function and the expansion involves only powers of $1/N$ (in fact, powers of $1/N^2$). This happens for the so-called "one-cut regime", and for the one matrix model, the existence of such an expansion has been proved for the one matrix model for real-valued, analytic potential [APS01], and then the coefficients are necessarily given by the symplectic invariants of the semiclassical spectral curve [Eyn04]. Beyond the one-cut regime, the Riemann-Hilbert steepest descent analysis [IKF90, DZ95] has been applied to find explicitly the asymptotics up to $o(1)$ in the one hermitian matrix model with real-valued, polynomial potential [BI99, DKM+99]. It features in general a pseudo-periodic behavior with N, encoded in the Theta function, but the one-cut regime can also be recovered with this Riemann-Hilbert method [EM03]. These results have later been extended to the one normal matrix model with complex-valued polynomial potential [BM09b]. This method can be used in principle to find recursively the subleading orders, although it does not allow to write the answer a priori explicitly to all orders. Definition 5.3 is expected to give the correct answer to all orders.

9.2.4 Baker-Akhiezer spinor kernel

The spinor kernel is related to the expectation value of ratios of characteristic polynomials:

$$\psi(z_1, z_2) = \left\langle \frac{\det(X(z_1) - M)}{\det(X(z_2) - M)} \right\rangle \frac{\sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)}}{X(z_1) - X(z_2)} \tag{9.19}$$

The dispersionless spinor kernel $\psi_{\text{cl}}(z_1, z_2)$ is the large N limit of $\psi(z_1, z_2)$. When sending $X(z_2) \to \infty$, one gets, after proper renormalization that the Baker-Akhiezer function is

$$\psi(z_1) = \text{"}\psi(z_1, \infty)\text{"} = \Big\langle \det(X(z_1) - M) \Big\rangle \tag{9.20}$$

which is clearly a polynomial in $X(z_1)$ of degree N. It is a classical result of random matrix theory [Meh04] that this is the orthogonal polynomial of degree N, orthogonal with respect to the measure $\mathrm{d}\mu$ for matrices of size 1. The dual Baker Akhiezer-function for matrix models is

$$\phi(z_1) = \text{"}\psi(\infty, z_1)\text{"} = \left\langle \frac{1}{\det(X(z_1) - M)} \right\rangle \tag{9.21}$$

It can be derived from orthogonality relations of the orthogonal polynomials that those spinor kernels satisfy Hirota equations, also called "determinantal formula", for instance:

$$\begin{aligned} & \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_3-x_4)} \left\langle \frac{\det(x_1-M)\,\det(x_2-M)}{\det(x_3-M)\,\det(x_4-M)} \right\rangle \\ = & \frac{1}{(x_1-x_3)(x_2-x_4)} \left\langle \frac{\det(x_1-M)}{\det(x_3-M)} \right\rangle \left\langle \frac{\det(x_2-M)}{\det(x_4-M)} \right\rangle \\ & - \frac{1}{(x_1-x_4)(x_2-x_3)} \left\langle \frac{\det(x_1-M)}{\det(x_4-M)} \right\rangle \left\langle \frac{\det(x_2-M)}{\det(x_3-M)} \right\rangle \end{aligned} \tag{9.22}$$

In that case, N is an integer (the size of the matrices), and Hirota equation is an equality between sequences indexed by N, not only of formal asymptotic series. This relations have been proved in [FS03] for the one matrix model, and in [Ber, AP04] for the two matrices model, from which the case of chain of matrices can be deduced. It shows that, modulo the heuristic derivation of asymptotics of matrix integrals, Hirota equation as stated as in Conjecture 7.4) hold for the semiclassical spectral curves of matrix models (however, not all algebraic curves fall in this class).

9.2.5 Differential systems

The orthogonal polynomials, as well as their duals, in all cases above, do satisfy some ODE of order d (d depends on the degrees of potentials). As we have seen in Section 8, the proof is a consequence of Hirota equations. For instance for the one matrix model, this is a second order ODE ($d = 2$). For the two matrices model with potentials V_1 and V_2, we always have $d = 1 + \deg V_2'$ (in case V_2' is a rational function, $\deg V_2'$ is the sum of degrees of all poles). In all cases we have a d-dimensional vector:

$$\vec{\Psi}(z) = \begin{pmatrix} \psi(z_1) \\ \vdots \end{pmatrix} \tag{9.23}$$

where the first entry is $\psi(z_1)$. The other entries are obtained from $\psi(z_1)$ with a procedure described in [BEH03a] and very similar to Section 3.4. This vector satisfies an ODE

$$\frac{\mathrm{d}}{\mathrm{d}\,X(z)}\,\vec{\Psi}(z) \;=\; \mathcal{D}(X(z))\;\vec{\Psi}(z) \tag{9.24}$$

where $\mathcal{D}(x)$ is a $d \times d$ matrix, whose entries are rational functions of x, that depend implicitly on N, on the coefficients of the potentials, and on the choice of integration contour Γ. The locus of eigenvalues of $\mathcal{D}(x)$, i.e. the polynomial equation

$$\mathcal{E}_N(x, y) = \det\left(y\,\mathrm{Id}_{d\times d} - \mathcal{D}(x)\right) = 0 \tag{9.25}$$

defines the finite N spectral curve. The semiclassical spectral curve corresponds to its large N limit:

$$\mathcal{E}(x, y) = \lim_{N\to\infty} \mathcal{E}_N(x, y). \tag{9.26}$$

There are also $d \times d$ differential systems for derivatives with respect to all coefficients of the potentials, and there is also a linear recursion relation on $N \to N+1$ (see [CI97] for the one matrix model). All these systems are compatible as shown in [BEH02] in full generality for the chain of matrices. Moreover, in all these matrix models, there is also a compatible Toda chain recursion equation relating Z_{N+1}, Z_N and Z_{N-1} [GMM+91]. We postpone to a future work the investigation of the presence of such relations in our construction.

9.2.6 Symplectic invariance

Notice that, in the 2-matrix model

$$Z = \int_{H_N(\Gamma)\times H_N(\tilde{\Gamma})} \mathrm{d}\mu(M_1, M_2),$$

we have defined our semiclassical spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$ from the large N limit of the resolvent $\mathcal{W}_1(x) = \langle \operatorname{tr}\frac{1}{x-M_1}\rangle$ associated to the matrix M_1. Since M_1 and M_2 play a symmetric role, it is clear that we would have obtained the same partition function, starting from the semiclassical spectral curve $\tilde{\mathcal{S}} = (\tilde{\mathcal{C}}, \tilde{X}, \tilde{Y})$ associated to the resolvent of matrix M_2, and thus we must have

$$\mathcal{T}(\mathcal{S}) = \mathcal{T}(\tilde{\mathcal{S}}). \tag{9.27}$$

One can easily find that the two spectral curves $\mathcal{S} = (\mathcal{C}, X, Y)$ and $\tilde{\mathcal{S}} = (\tilde{\mathcal{C}}, \tilde{X}, \tilde{Y})$ are related by $\mathcal{C} = \tilde{\mathcal{C}}$ and $\tilde{X} = Y$, $\tilde{Y} = X$, in other words they are symplectically equivalent, and the fact that $\mathcal{T}(\mathcal{S}) = \mathcal{T}(\tilde{\mathcal{S}})$ can be seen as a consequence of the symplectic invariance of the F_g's.

In fact, this is a manifestation at large N of an exact result for finite N. The orthogonal polynomials $\vec{\Psi}(x)$ associated to matrix M_1 satisfy an ODE of some order d (see Eqn. 9.23):

$$\frac{\mathrm{d}}{\mathrm{d}\,X(z)}\,\vec{\Psi}(z) \;=\; \mathcal{D}(X(z))\;\vec{\Psi}(z) \tag{9.28}$$

whereas the orthogonal polynomials $\vec{\tilde{\Psi}}(y)$ associated to matrix M_2 satisfy another ODE of some order $\tilde{d}$ (in general $\tilde{d} \neq d$):

$$\frac{\mathrm{d}}{\mathrm{d}\,Y(z)}\,\vec{\tilde{\Psi}}(z) \;=\; \tilde{\mathcal{D}}(Y(z))\;\vec{\tilde{\Psi}}(z) \tag{9.29}$$

It was discovered in [BEH03b] that:

$$\det\left(y\,\mathrm{Id}_{d\times d} - \mathcal{D}(x)\right) = \det\left(x\,\mathrm{Id}_{\tilde{d}\times\tilde{d}} - \tilde{\mathcal{D}}(y)\right). \tag{9.30}$$

This also implies that the semiclassical spectral curve defined from M_1 or from M_2 are related as explained above.

10 Conclusion

For integrable systems with a small dispersive parameter $1/N$, using the theory of symplectic invariants [EO07a], we have introduced a formal object $\mathcal{T}$, which is conjectured to be a Tau function in the sense that it satisfies Hirota equations. It is challenging to find a full proof of Conjecture 7.4.

This construction was strongly motivated by results known for matrix models, but developed independently. In some sense, it gives the answer to all order in $1/N$ of a Whitham averaging. All properties mentioned in this article, correspond to well-known or conjectured properties of matrix models. Our construction is thus an axiomatized extension of the properties of matrix models, to arbitrary algebraic spectral curves. One can wonder how to generalize the construction. First, to algebraic curves in $\mathbb{C}^* \times \mathbb{C}^*$, having in view the mirror curve appearing in Gromov-Witten theory, which all are of the form $\mathrm{Pol}(e^x, e^y) = 0$. Second, to the bundles appearing in generalized matrix models like the $O(n)$ model, which may be related to isomonodromic deformations on Riemann surfaces of positive genus and to Hitchin systems [Hit87]. And, even further to $\mathcal{D}$-modules, for which an adapted topological recursion (the so-called β-deformation of the topological recursion) has been found with similiar properties [CEM09, CEM10].

Eventually, it remains to compare this construction with other approaches (Frobenius manifolds, Poisson bracket structures, Segal-Wilson formalism in the Grassmannian [SW85], etc.), study its consequences and better understand the underlying geometry.

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A Proof of Conjecture 7.1 up to $o(1/N)$

Proposition A.1 *$\psi(z_1,z_2)$ is self-replicating at least up to $o(1/N)$:*

$$\frac{1}{N}\,\delta_z\psi(z_1,z_2)+\psi(z_1,z)\psi(z,z_2)=o(1/N)$$

Proof. Let us start from $\psi_{12}=\psi(z_1,z_2)$ written as

$$\psi_{12}=\frac{\mathrm{e}^{N\int_2^1 Y\mathrm{d}X}}{E_{12}}\,\frac{\Theta_{12}}{\Theta}\left\{1+\frac{1}{N}\hat{\psi}_{12}+o(1/N)\right\}\tag{A.1}$$

with

$$\begin{aligned}\hat{\psi}_{12} \;=\; & \int_2^1\omega_1^{(1)}+\frac{1}{6}\int_2^1\int_2^1\int_2^1\omega_3^{(0)}\\ & +\frac{1}{2}\frac{\Theta''_{12}}{\Theta_{12}}\oint\oint\int_2^1\omega_3^{(0)}+\frac{1}{2}\frac{\Theta'_{12}}{\Theta_{12}}\oint\int_2^1\int_2^1\omega_3^{(0)}\\ & +\Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)F'_1+\frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}}-\frac{\Theta'''}{\Theta}\Big)F'''_0\end{aligned}\tag{A.2}$$

where, to shorten notations, 1 means z_1, 2 means z_2, $\oint$ means the contour integral around $\mathcal{B}$-cycles (indices are understood in tensor notations, i.e. contracted with the indices of derivatives of Θ), and Θ_{12} means

$$\Theta_{12}=\Theta(\mathbf{w}_0+2i\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\,|\,\tau),\qquad\Theta=\Theta(\mathbf{w}_0\,|\,\tau)\tag{A.3}$$

We remind that (Eqn. 5.3):

$$\omega_3^{(0)}(z_0,z_1,z_2)=\sum_i\operatorname*{Res}_{z\to a_i}\frac{B(z_0,z)B(z_1,z)B(z_2,z)}{\mathrm{d}X(z)\,\mathrm{d}Y(z)}\tag{A.4}$$

and by special geometry:

$$F'''_0=\oint\oint\oint\omega_3^{(0)}=\sum_i\operatorname*{Res}_{z\to a_i}\frac{(\mathrm{d}\mathbf{v}(z))^3}{\mathrm{d}X(z)\,\mathrm{d}Y(z)}\tag{A.5}$$

where we have denoted for short

$$\mathrm{d}\mathbf{v}(z)=2i\pi\,\mathrm{d}\mathbf{u}(z)=\oint_{\mathcal{B}}B(z,\cdot)\tag{A.6}$$

The expression for $\hat{\psi}_{12}$ is thus:

$$\begin{aligned}
\hat{\psi}_{12} &= \int_2^1 \omega_1^{(1)} + \frac{1}{6}\int_2^1\int_2^1\int_2^1 \omega_3^{(0)} \\
&+\frac{1}{2}\frac{\Theta''_{12}}{\Theta_{12}}\oint\oint\int_2^1 \omega_3^{(0)} + \frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\oint\int_2^1\int_2^1 \omega_3^{(0)} \\
&+\Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)F'_1 + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)F'''_0 \\
&= \int_2^1 \omega_1^{(1)} + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)F'_1 \\
&+\sum_i \operatorname*{Res}_{z\to a_i} \frac{1}{\mathrm{d}X(z)\,\mathrm{d}Y(z)}\Big\{\frac{1}{6}(\mathrm{d}S_{12}(z))^3 + \frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z)(\mathrm{d}S_{12}(z))^2 \\
&+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)(\mathrm{d}\mathbf{v}(z))^2\mathrm{d}S_{12}(z) + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z))^3\Big\}
\end{aligned} \tag{A.7}$$

We need to apply the insertion operator $\frac{1}{N}\,\delta_z$ to $\hat{\psi}_{12}$, and obtain the result up to $o(1)$. Only the variation of NF'_0 appearing in the Theta functions contributes to this order:

$$\begin{aligned}
\frac{1}{N}\,\delta_z\hat{\psi}_{12} &= \Big[\Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)' F'_1 \\
&+\sum_i \operatorname*{Res}_{z'\to a_i} \frac{1}{\mathrm{d}X(z')\,\mathrm{d}Y(z')}\Big\{\frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2 \\
&+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^2\mathrm{d}S_{12}(z') + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\Big\}\Big]\mathrm{d}\mathbf{v}(z) \\
&+o(1)
\end{aligned} \tag{A.8}$$

This allows us to compute:

$$\begin{aligned}
&\frac{1}{N}\,\delta_z \ln\psi_{12} \\
&= \mathrm{d}S_{12}(z) + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)\mathrm{d}\mathbf{v}(z) \\
&+\frac{1}{2N}\int_2^1\int_2^1 \omega_3^{(0)}(z,\cdot,\cdot) + \frac{1}{N}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\oint\int_2^1 \omega_3^{(0)}(z,\cdot,\cdot) \\
&+\frac{1}{2N}\Big(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\Big)\oint\oint \omega_3^{(0)}(z,\cdot,\cdot) + \frac{1}{N^2}\,\delta_z\hat{\psi}_{12} + o(1/N) \\
&= \mathrm{d}S_{12}(z) + \mathrm{d}\mathbf{v}(z)\,\Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big) \\
&+\frac{1}{N}\sum_i \operatorname*{Res}_{z'\to a_i} \frac{B(z',z)}{\mathrm{d}X(z')\,\mathrm{d}Y(z')}\Big\{\frac{1}{2}(\mathrm{d}S_{12}(z))^2 + \Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') \\
&+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2\Big\} + \frac{1}{N^2}\,\delta_z\hat{\psi}_{12} + o(1/N) \\
&= \mathrm{d}S_{12}(z) + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)\mathrm{d}\mathbf{v}(z) + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)' F'_1\,\mathrm{d}\mathbf{v}(z)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N}\sum_i \operatorname*{Res}_{z'\to a_i} \frac{1}{\mathrm{d}X(z')\,\mathrm{d}Y(z')}\Big\{\frac{1}{2}(\mathrm{d}S_{12}(z'))^2 B(z',z) + \Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z',z)\\
&+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}-\frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2 B(z',z) + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^2\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z)\\
&+\frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2\mathrm{d}\mathbf{v}(z) + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}}-\frac{\Theta'''}{\Theta}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z)\Big\}\\
&+o(1/N)
\end{aligned}
\tag{A.9}
$$

We may transform the first line using the refined duality equation established in Theorem 3.3, which is a consequence of the Fay identity satisfied by the Theta function of the spectral curve. It can be rephrased as:

$$
\mathrm{d}S_{12}(z) + \Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)\mathrm{d}\mathbf{v}(z) = -\frac{E_{12}}{E_{1z}E_{z2}}\ \frac{\Theta_{1z}\Theta_{z2}}{\Theta\,\Theta_{12}}
\tag{A.10}
$$

On the other hand we have

$$
\begin{aligned}
&\frac{\psi_{1z}\,\psi_{z2}}{\psi_{12}}\\
=\ &\frac{E_{12}}{E_{1z}E_{z2}}\ \frac{\Theta_{1z}\Theta_{z2}}{\Theta\,\Theta_{12}}\Big(1+\frac{1}{N}(\hat\psi_{1z}+\hat\psi_{z2}-\hat\psi_{12})+o(1/N)\Big)\\
=\ &\frac{E_{12}}{E_{1z}E_{z2}}\ \frac{\Theta_{1z}\Theta_{z2}}{\Theta\,\Theta_{12}}\Big(1+\frac{1}{N}\Big[\frac{\Theta'_{1z}}{\Theta_{1z}}+\frac{\Theta'_{z2}}{\Theta_{z2}}-\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big]\,F'_1\\
&+\frac{1}{N}\sum_i \operatorname*{Res}_{z'\to a_i}\frac{1}{\mathrm{d}X(z')\,\mathrm{d}Y(z')}\Big\{\frac{1}{2}\Big(\frac{\Theta'_{1z}}{\Theta_{1z}}(\mathrm{d}S_{1z}(z'))^2+\frac{\Theta'_{z2}}{\Theta_{z2}}(\mathrm{d}S_{z2}(z'))^2-\frac{\Theta'_{12}}{\Theta_{12}}(\mathrm{d}S_{12}(z'))^2\Big)\mathrm{d}\mathbf{v}(z')\\
&+\frac{1}{2}\Big(\frac{\Theta''_{1z}}{\Theta_{1z}}\mathrm{d}S_{1z}(z')+\frac{\Theta''_{z2}}{\Theta_{z2}}\mathrm{d}S_{z2}(z')-\frac{\Theta''_{12}}{\Theta_{12}}\mathrm{d}S_{12}(z')\Big)(\mathrm{d}\mathbf{v}(z'))^2\\
&+\frac{1}{6}\Big(\frac{\Theta'''_{1z}}{\Theta_{1z}}+\frac{\Theta'''_{z2}}{\Theta_{z2}}-\frac{\Theta'''_{12}}{\Theta_{12}}-\frac{\Theta'''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^3\Big\}-\frac{1}{2}\mathrm{d}S_{1z}(z')\mathrm{d}S_{z2}(z')\mathrm{d}S_{12}(z')+o(1/N)\Big)
\end{aligned}
\tag{A.11}
$$

Let us now compute:

$$
\begin{aligned}
&\frac{1}{N}\,\delta_z\ln\psi_{12}+\frac{\psi_{1z}\,\psi_{z2}}{\psi_{12}}\\
=\ &\Big\{\Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)'\mathrm{d}\mathbf{v}(z)+\frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\,\Theta_{12}}\Big(\frac{\Theta'_{1z}}{\Theta_{1z}}+\frac{\Theta'_{z2}}{\Theta_{z2}}-\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)\Big\}F'_1\\
&+\sum_i \operatorname*{Res}_{z'\to a_i}\frac{1}{\mathrm{d}X(z')\,\mathrm{d}Y(z')}\Big\{\frac{1}{2}(\mathrm{d}S_{12}(z'))^2B(z',z)+\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z',z)\\
&+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}-\frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2B(z',z)+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^2\mathrm{d}S_{12}(z')\mathrm{d}\mathbf{v}(z)\\
&+\frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2\mathrm{d}\mathbf{v}(z)+\frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}}-\frac{\Theta'''}{\Theta}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z)\\
&+\frac{E_{12}}{E_{1z}E_{z2}}\ \frac{\Theta_{1z}\Theta_{z2}}{\Theta\,\Theta_{12}}\ \Big[\frac{1}{2}\Big(\frac{\Theta'_{1z}}{\Theta_{1z}}(\mathrm{d}S_{1z}(z'))^2+\frac{\Theta'_{z2}}{\Theta_{z2}}(\mathrm{d}S_{z2}(z'))^2-\frac{\Theta'_{12}}{\Theta_{12}}(\mathrm{d}S_{12}(z'))^2\Big)\mathrm{d}\mathbf{v}(z')
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Big(\frac{\Theta''_{1z}}{\Theta_{1z}}\mathrm{d}S_{1z}(z')+\frac{\Theta''_{z2}}{\Theta_{z2}}\mathrm{d}S_{z2}(z')-\frac{\Theta''_{12}}{\Theta_{12}}\mathrm{d}S_{12}(z')\Big)(\mathrm{d}\mathbf{v}(z'))^2\\
&+\frac{1}{6}\Big(\frac{\Theta'''_{1z}}{\Theta_{1z}}+\frac{\Theta'''_{z2}}{\Theta_{z2}}-\frac{\Theta'''_{12}}{\Theta_{12}}-\frac{\Theta'''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^3-\frac{1}{2}\mathrm{d}S_{1z}(z')\mathrm{d}S_{z2}(z')\mathrm{d}S_{12}(z')\Big]\Big\}\\
&+o(1/N) \qquad \text{(A.12)}
\end{aligned}
$$

The coefficient of F'_1 vanishes, as we can see by computing the gradient of Eqn. A.10 to leading order in N (recall that N enters in the definition of our Θ through the point $\mathbf{w}_0 = NF'_0$):

$$
\begin{aligned}
&\Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)'\mathrm{d}\mathbf{v}(z)\\
&= -\frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{2z}}{\Theta\Theta_{12}}\Big(\frac{\Theta'_{1z}}{\Theta_{1z}}+\frac{\Theta'_{z2}}{\Theta_{z2}}-\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big) \qquad \text{(A.13)}
\end{aligned}
$$

This identity can also be seen as a consequence of Fay identity.

Let us now study the residue term of Eqn. A.12, which we write

$$
\sum_i \operatorname*{Res}_{z'\to a_i}\frac{H_{12}(z',z)}{\mathrm{d}X(z')\,\mathrm{d}Y(z')} \qquad \text{(A.14)}
$$

First, notice that by construction, $H_{12}(z',z)$ is a meromorphic 1-form in the variable z, which means that it has trivial monodromy when z goes around a non-trivial cycle. It may have simple poles at $z=z_1$ or $z=z_2$ coming from the ratio of prime forms, but the expression in $[\cdots]$ vanish when $z=z_1$ or z_2, so $H_{12}(z,z')$ is actually regular at $z=z_1$ or $z=z_2$. It may also have a singularity at $z=z'$ coming from the term $[\cdots]$, which is atmost a double pole. To leading order when $z\to z'$, we find:

$$
\begin{aligned}
H_{12}(z',z) &= \frac{1}{2}\mathrm{d}S_{12}(z')^2+\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}-\frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2\\
&\quad \frac{E_{12}}{E_{1z'}E_{z'2}}\frac{\Theta_{1z'}\Theta_{z'2}}{\Theta\Theta_{12}}\Big\{\frac{1}{2}\Big(\frac{\Theta'_{1z'}}{\Theta_{1z}}+\frac{\Theta'_{z'2}}{\Theta_{z'2}}\Big)\mathrm{d}\mathbf{v}(z')+\frac{1}{2}\mathrm{d}S_{12}(z')\Big\}\\
&\quad +O(\xi_{z'}^{-1}(z)) \qquad \text{(A.15)}
\end{aligned}
$$

where $\xi_{z'}$ is our notation for a local coordinate centered at z'. We use again Eqn. A.10 and find:

$$
\begin{aligned}
H_{12}(z',z) &= \frac{1}{2}\mathrm{d}S_{12}(z')^2+\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')+\frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}-\frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2\\
&\quad -\Big(\mathrm{d}S_{12}(z')+\Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)\mathrm{d}\mathbf{v}(z)\Big)\Big\{\frac{1}{2}\Big(\frac{\Theta'_{1z'}}{\Theta_{1z}}+\frac{\Theta'_{z'2}}{\Theta_{z'2}}\Big)\mathrm{d}\mathbf{v}(z')+\frac{1}{2}\mathrm{d}S_{12}(z')\Big\}\\
&\quad +O(\xi_{z'}^{-1}(z))\\
&= \Big\{-\mathrm{d}S_{12}(z')\Big(-\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}+\frac{\Theta'_{1z'}}{\Theta_{1z'}}+\frac{\Theta'_{z'2}}{\Theta_{z'2}}\Big)\\
&\quad -\Big(\frac{\Theta'_{12}}{\Theta_{12}}-\frac{\Theta'}{\Theta}\Big)\Big(\frac{\Theta'_{1z'}}{\Theta_{1z'}}+\frac{\Theta'_{z'2}}{\Theta_{z'2}}\Big)\mathrm{d}\mathbf{v}(z')+\Big(\frac{\Theta''_{12}}{\Theta_{12}}-\frac{\Theta''}{\Theta}\Big)\mathrm{d}\mathbf{v}(z')\Big\}\frac{\mathrm{d}\mathbf{v}(z')}{2}\\
&\quad +O(\xi_{z'}^{-1}(z)) \qquad \text{(A.16)}
\end{aligned}
$$

We can rearrange the terms:

$$\begin{aligned} H_{12}(z',z) &= \Big\{ -\Big[\mathrm{d}S_{12}(z') + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)\mathrm{d}\mathbf{v}(z')\Big] \Big(\frac{\Theta'_{1z'}}{\Theta_{1z'}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}} - \frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big) \\ &\quad + \Big(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\Big)' \mathrm{d}\mathbf{v}(z') \Big\} \frac{\mathrm{d}\mathbf{v}(z')}{2} + O(\xi_{z'}^{-1}(z)) \end{aligned}$$

and according to Eqn. A.13, we see that $H_{12}(z',z) \in O(\xi_{z'}^{-1}(z))$. Therefore, $H_{12}(z',z)$ is a meromorphic function whose only singularity is a pole atmost simple at $z = z'$. But a meromorphic function cannot have a single simple pole, so $H_{12}(z,z')$ must be holomorphic, and we can write it:

$$H_{12}(z,z') = h_{12}(z')\,\mathrm{d}\mathbf{v}(z) \tag{A.17}$$

Since the prefactor $h_{12}(z')$ is independent of z, we may compute it by specializing to $z = z_1$ in $H_{12}(z,z_1)$ defined from Eqn. A.12. Doing so, we obtain:

$$\begin{aligned} h_{12}(z')\,\mathrm{d}\mathbf{v}(z_1) &= \frac{1}{2}(\mathrm{d}S_{12}(z'))^2 B(z',z_1) + \Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z',z_1) + \\ &\quad + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2 B(z',z_1) + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z_1) \\ &\quad \frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2\mathrm{d}\mathbf{v}(z_1) + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z_1) \\ &\quad - \mathrm{d}_{z=z_1}\Big[\frac{1}{2}\Big(\frac{\Theta'_{1z}}{\Theta_{1z}}(\mathrm{d}S_{1z}(z'))^2 + \frac{\Theta'_{z2}}{\Theta_{z2}}(\mathrm{d}S_{z2}(z'))^2 - \frac{\Theta'_{12}}{\Theta_{12}}(\mathrm{d}S_{12}(z'))^2\Big)\mathrm{d}\mathbf{v}(z') \\ &\quad + \frac{1}{2}\Big(\frac{\Theta''_{1z}}{\Theta_{1z}}\mathrm{d}S_{1z}(z') + \frac{\Theta''_{z2}}{\Theta_{z2}}\mathrm{d}S_{z2}(z') - \frac{\Theta''_{12}}{\Theta_{12}}\mathrm{d}S_{12}(z')\Big)(\mathrm{d}\mathbf{v}(z'))^2 \\ &\quad + \frac{1}{6}\Big(\frac{\Theta'''_{1z}}{\Theta_{1z}} + \frac{\Theta'''_{z2}}{\Theta_{z2}} - \frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^3 - \frac{1}{2}\mathrm{d}S_{1z}(z')\mathrm{d}S_{z2}(z')\mathrm{d}S_{12}(z')\Big] \\ &= \frac{1}{2}(\mathrm{d}S_{12}(z'))^2 B(z',z_1) + \Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z',z_1) + \\ &\quad + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2 B(z',z_1) + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z_1) \\ &\quad \frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2\mathrm{d}\mathbf{v}(z_1) + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)'(\mathrm{d}\mathbf{v}(z'))^3\mathrm{d}\mathbf{v}(z_1) \\ &\quad - \Big[\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)B(z_1,z')\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') + \frac{1}{2}\Big(\frac{\Theta'_{12}}{\Theta_{12}}\Big)'\mathrm{d}\mathbf{v}(z')\mathrm{d}\mathbf{v}(z_1)(\mathrm{d}S_{12}(z'))^2 \\ &\quad \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\Big)(\mathrm{d}\mathbf{v}(z'))^2 B(z_1,z') + \frac{1}{2}\Big(\frac{\Theta''_{12}}{\Theta_{12}}\Big)'(\mathrm{d}\mathbf{v}(z'))^2\mathrm{d}\mathbf{v}(z_1)\mathrm{d}S_{12}(z') \\ &\quad + \frac{1}{6}\Big(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\Big)'(\mathbf{d}v(z'))^3\mathrm{d}\mathbf{v}(z_1) + \frac{1}{2}B(z_1,z')(\mathrm{d}S_{12}(z'))^2\Big] \\ &= 0 \end{aligned} \tag{A.18}$$

All the terms eventually cancel each other. Thus $H_{12}(z, z') = 0$, and coming back to Eqn. A.12, this proves:

$$\frac{1}{N}\,\delta_z \ln \psi(z_1, z_2) + \frac{\psi(z_1, z)\psi(z, z_2)}{\psi(z_1, z_2)} = o(1/N) \tag{A.19}$$

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